Triangle equalities

Whenever inequalities arise in mathematics, it is natural and often important to know when the two sides of the inequality are equal. For example, in the Schwarz Inequality for inner product spaces, equality holds if and only if the two vectors in the expression are linearly dependent (either or both are zero, or if both are nonzero then one is a multiple of the other. Of course, one can ask similar questions regarding the Triangle Equality for vector lengths and the related inequality for distances between two points in an inner product space. In each case there is a simple answer:

1. In the case of the Triangle Equality for vector lengths, equality holds if and only if one vector is a nonnegative multiple of the other.

2. In the case of the Triangle Inequality for distances in an inner product space (***i.e.***, the inequality $| x - z | \leq | x - y | + | y - z |$), equality holds if and only if the vector y lies on the closed line segment joining x to z. In linear-algebraic terms this means that $y = t\,x + (1 - t)\,z$ for some t in the unit interval [0,1].

Derivations for these results appear on pages 5 and 6 of the following:

http://math.ucr.edu/~res/math133/geometrynotes01.f13.pdf