$$= \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2$$
$$\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2$$
$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

Taking the square roots yields the Triangle Inequality for vector lengths.■

The following consequences of the triangle inequalities are often useful; they give lower bounds instead of upper bounds:

Proposition 1. *For all vectors* **a**, **b**, **c**, x, y *we have the following*:

$$\big|\, ||x|| - ||y||\, \big| \quad \leq \quad ||x - y||$$

$$|d(\mathbf{b}, \mathbf{c}) - d(\mathbf{c}, \mathbf{a})| \quad \leq \quad d(\mathbf{a}, \mathbf{b})$$

Proof. As above, the main point is to verify the first inequality and to derive the second from it. Two applications of the Triangle Inequality for vector lengths show that

$$||x|| \quad \leq \quad ||x - y|| + ||y||$$

$$||y|| \quad \leq \quad ||x|| + ||y - x|| \quad = \quad ||x|| + ||x - y||$$

and these may be rewritten as follows:

$$||x|| - ||y|| \quad \leq \quad ||x - y||\,, \qquad ||y|| - ||x|| \quad \leq \quad ||x - y||$$

These are equivalent to the first inequality in the proposition, and the second follows by making the substitutions $y = \mathbf{c} - \mathbf{a}$ and $x = \mathbf{b} - \mathbf{c}$ in the first one. Of course, by the symmetry properties of distance and the expression $\big|\, ||x|| - ||y||\, \big|$ this inequality can be rewritten many ways, including $|d(\mathbf{a}, \mathbf{c}) - d(\mathbf{b}, \mathbf{c})| \leq d(\mathbf{a}, \mathbf{b})$.■

Equality in the Triangle Inequalities

The preceding material is covered in many linear algebra courses, but we shall now go one step beyond such courses. As already noted, one has an equality

$$|\langle x, y\rangle| \quad = \quad ||x|| \cdot ||y||$$

corresponding to the Cauchy – Schwarz Inequality if and only if the vectors x and y are linearly dependent. For our purposes it will be important to know the analogous conditions under which one has the equations

$$||x + y|| \quad = \quad ||x|| + ||y|| \qquad d(\mathbf{a}, \mathbf{b}) \quad = \quad d(\mathbf{a}, \mathbf{c}) + d(\mathbf{c}, \mathbf{b})$$

associated to the Triangle Inequalities.

Proposition 2. *Two nonzero vectors* x *and* y *satisfy* $||x + y|| = ||x|| + ||y||$ *if and only if each is a nonnegative multiple of the other*.

Proposition 3. *Three vectors* **a**, **b** *and* **c** *satisfy* $d(\mathbf{a}, \mathbf{b}) = d(\mathbf{a}, \mathbf{c}) + d(\mathbf{c}, \mathbf{b})$ *if and only if* $\mathbf{c} - \mathbf{a} = s(\mathbf{b} - \mathbf{a})$, *where* $0 \leq s \leq 1$.

Proof of Proposition 2. If one of the vectors is nonzero, then it is clear that equality holds and that one of the vectors is a nonnegative multiple of the other, and this is why we assume that both x and y are nonzero.

If we look back at the derivation of the Triangle Inequality, we see that the crucial step in deriving an inequality comes from applying the Cauchy – Schwarz Inequality to x and y. In particular, it follows that $||x + y|| = ||x|| + ||y||$ will hold if and only if we have

$$\langle x, y \rangle = ||x|| \, ||y||$$

(note that this condition is stronger than equality in the Cauchy – Schwarz Inequality, for we the latter involves the absolute value of the inner product and not the inner product itself). It will suffice to show that this stronger equation holds if and only if the nonzero vectors x and y are positive multiples of each other. If $y = tx$ where t is positive, then we have

$$\langle x, y \rangle = \langle x, tx \rangle = t\,||x||^2 = ||x||\,||tx|| = ||x||\,||y||$$

which shows the "if" direction. Conversely, if equality holds then the conclusion of the Cauchy – Schwarz Inequality shows that x and y are nonzero multiples of each other, so let $y = tx$ where t is nonzero; we need to show that t is positive. But now we have

$$t\,||x||^2 = \langle x, tx \rangle = \langle x, y \rangle = ||x||\,||y|| =$$
$$||x||\,||tx|| = |t|\,||x||^2$$

which implies that $t = |t|$ and hence that t is positive.■

Proof of Proposition 3. Suppose first that **a** = **b**. Then we have $0 = d$(**a**, **b**) $= d$(**a**, **c**) $+ d$(**c**, **b**) if and only if both summands on the right hand side are equal to zero, which is equivalent to **a** = **b** = **c**, so that the conclusion is true for trivial reasons.

Suppose now that **a** and **b** are unequal. By Proposition 1 and the definition of distance, we know that d(**a**, **b**) $= d$(**a**, **c**) $+ d$(**c**, **b**) if and only if either (1) one of **a** = **c** or **b** = **c** is true, (2) the vectors **c** − **a** and **b** − **c** are positive multiples of each other. In the first cases we have (respectively) either **c** − **a** $= 0 \cdot$(**b** − **a**) or **c** − **a** $= 1 \cdot$(**b** − **a**), so the conclusion is true if either of **a** = **c** or **b** = **c** is true. Thus we are left with the case where **b** − **c** $= t$(**c** − **a**) for some $t > 0$. We then have **b** − **a** = (**c** − **a**) + (**b** − **c**) $= (1 + t) \cdot$(**c** − **a**), so that

$$\mathbf{c} - \mathbf{a} = (1 + t)^{-1} \cdot (\mathbf{b} - \mathbf{a})$$

for some positive scalar t. To conclude the argument, note that the latter is equivalent to **c** − **a** $= s$(**b** − **a**) for some scalar s satisfying the conditions $0 < s < 1$.■