Diagonalizing Symmetric Matrices: Definition & Examples

In this lesson, we define symmetric and diagonal matrices. We then use eigenvalues and eigenvectors to form a very special matrix which is then used to diagonalize a symmetric matrix.

Getting to There from Here

Symmetric matrices appear often in math, science and engineering. In this lesson, we start with a symmetric matrix and show how to get a diagonal matrix. But first, some definitions.

Some Definitions

A **diagonal matrix**, D, has numbers along the main diagonal and zeros everywhere else. A **symmetric matrix**, A, has equal numbers in the off-diagonal locations.

The task is to find a matrix P which will let us convert A into D.

Once we get the matrix P, then $D = P^t AP$. The **transpose of P** is written as P^t.

This is a lot of terminology to absorb all at once. Let's work through the process step-by-step with actual examples of finding P and P^t.

The Steps for Diagonalizing a Symmetric Matrix

Step 1: Find the eigenvalues of A.

Here's a typical symmetric matrix:

$$A = \begin{bmatrix} 6.8 & 2.4 \\ 2.4 & 8.2 \end{bmatrix}$$

See the same number, 2.4, in the off-diagonal locations?

We are going to play with the equation A - λI.

For now, think of λ (lambda) as being a variable like *x*. And the ''I'' matrix is the **identity matrix** which is a special diagonal matrix having 1's along the main diagonal.

Substituting for A and I in A - λI:

$$A - \lambda I = \begin{bmatrix} 6.8 & 2.4 \\ 2.4 & 8.2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiply through the I matrix by λ

$$A - \lambda I = \begin{bmatrix} 6.8 & 2.4 \\ 2.4 & 8.2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

Subtract the two matrices

$$A - \lambda I = \begin{bmatrix} 6.8 - \lambda & 2.4 \\ 2.4 & 8.2 - \lambda \end{bmatrix}$$

Take the determinant of the resulting matrix

$$\det(A - \lambda I) = (6.8 - \lambda)(8.2 - \lambda) - 2.4^2$$

Expand the two factors enclosed in parentheses on the right-hand side.

$$\det(A\text{-}\lambda I) = 6.8(8.2) + \lambda^2 - \lambda(6.8 + 8.2) - 2.4^2$$

Continue to simplify

$$\det(A - \lambda I) = 55.76 + \lambda^2 - \lambda(15) - 5.76$$

The right-hand-side is almost ready to be factored. Just reorganize the terms.

$$\det(A - \lambda I) = \lambda^2 - 15\lambda + 50$$

This is factored as

$$\det(A - \lambda I) = (\lambda - 10)(\lambda - 5)$$

Now, we set det(A - λI) to 0 and solve for λ. We get

$$(\lambda - 10)(\lambda - 5) = 0$$

Either of the factors (λ - 10) or (λ - 5) could be zero. If (λ - 10) = 0, then λ = 10. We call this λ_1. The other possibility is (λ - 5) = 0 which means $\lambda_2 = 5$. The λ_1 and λ_2 are the **eigenvalues** of A.

Step 2: Find the eigenvectors.

A matrix has dimensions. This is the number of rows and number of columns. For example, a 3x2 matrix has 3 rows and 2 columns. The matrix, A, is a 2x2 matrix. If either the number of rows or the number of columns of a matrix is one, we call this matrix a **vector**. The vectors we will use have 2 rows and 1 column.

What if multiplying a matrix by a certain vector gives the same result as multiplying this vector by an eigenvalue? This special vector is called an eigenvector. We are looking for the eigenvector, v_1, which goes with the eigenvector, λ_1. The words ''which goes with'' are commonly replaced with ''associated with''. In other words, we are looking for the eigenvector, v_1, associated with the eigenvalue, λ_1, satisfying

$$Av_1 = \lambda_1 v_1$$

For now, we don't know the numbers in v_1. No problem. We will use the letters *a* and *b*.

Substituting

$$\begin{bmatrix} 6.8 & 2.4 \\ 2.4 & 8.2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 10 \begin{bmatrix} a \\ b \end{bmatrix}$$

Multiplying the matrix times the vector produces two equations. The first equation is

$$6.8a + 2.4b = 10a$$

Bringing all terms to the left:

$$6.8a + 2.4b - 10a = 0$$

Simplifying

$$-3.2a + 2.4b = 0$$

The second of the two equations is

$$2.4a + 8.2b = 10b$$

Bringing all the terms to the left-hand-side

$$2.4a + 8.2b - 10b = 0$$

Simplifying

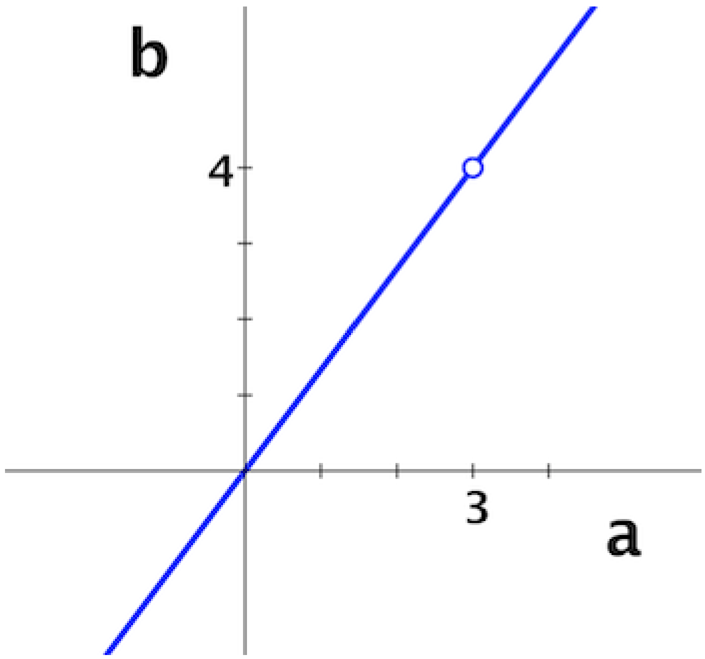
$$2.4a - 1.8b = 0$$

Multiplying the matrix times the vector gave us two equations:

- $-3.2a + 2.4b = 0$

- 2.4*a* - 1.8*b* = 0

Plotting *b* vs *a* gives a straight line for each equation. And, the straight lines are the same straight line! Unlike two lines crossing at one point giving a unique solution for *a* and *b*, these lines have an infinite number of points in common. The best we can do is to select one of the points and use it to relate *a* and *b*.



a=3 and b=4 solves both equations

Try substituting 3 for *a* and 4 for *b* in each equation to verify these numbers work. Thus, the eigenvector is

$$v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Note, as a practical matter, we could have chosen any point on the line other than the point at the origin. The numbers 3 and 4 are nice because they are whole numbers. But we could have let *a* = 1 which would give *b* = 4/3. Both equations are satisfied with this choice as well. Later we will normalize the eigenvector. The normalized eigenvector is unique regardless of which point we choose on the line. The point at the origin provides no information because it says zero times any number is a solution.

To find the other eigenvector, use the second eigenvalue.

$$Av_2 = \lambda_2 v_2$$

As before, we substitute for A and λ with the idea of finding the numbers for the eigenvector, v_2.

$$\begin{bmatrix} 6.8 & 2.4 \\ 2.4 & 8.2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 5 \begin{bmatrix} a \\ b \end{bmatrix}$$

As before, we get two equations and simplify. The first result is

$$6.8a + 2.4b = 5a \Rightarrow 1.8a + 2.4b = 0$$

and the second is

$$2.4a + 8.2b = 5b \Rightarrow 2.4a + 3.2b = 0$$

Once again, we have two equations with no unique answer. Two values that work are a = -4 and b = 3.

Thus, the eigenvector, associated with λ = 5 is

$$v_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

Step 3: Normalize the eigenvectors.

Next, we make the length of each eigenvector equal to 1. This is called **normalizing**.

We find the length of the vector, v_1, by taking the square root of the sum of 3 squared and 4 squared.

$$\begin{aligned} ||v_1|| &= \sqrt{3^2 + 4^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

v_1 surrounded by a pair of vertical lines means ''the length of v_1''.

The length of v_1 is 5.

To normalize v_1, we divide v_1 by its length.

$$u_1 = \frac{v_1}{||v_1||}$$

Just to be clear, the normalized version of v_1 is written as u_1.

Substituting we get

$$u_1 = \frac{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}{5}$$

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