7.7 GRAM–SCHMIDT ORTHOGONALIZATION PROCESS

Suppose $\{v_1, v_2, \ldots, v_n\}$ is a basis of an inner product space V. One can use this basis to construct an orthogonal basis $\{w_1, w_2, \ldots, w_n\}$ of V as follows. Set

$$w_1 = v_1$$
$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$
$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$w_n = v_n - 1\frac{\langle v_n, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_n, w_2\rangle}{\langle w_2, w_2\rangle} w_2 - \ldots - \frac{\langle v_n, w_{n-1}\rangle}{\langle w_{n-1}, w_{n-1}\rangle} w_{n-1}$$

In other words, for $k = 2, 3, \ldots, n$, we define

$$w_k = v_k - c_{k1}w_1 - c_{k2}w_2 - \ldots - c_{k,k-1}w_{k-1}$$

where $c_{ki} = \langle v_k, w_i\rangle / \langle w_i, w_i\rangle$ is the component of v_k along w_i. By Theorem 7.8, each w_k is orthogonal to the preceeding w's. Thus $w_1, w_2, \ldots, w_n$ form an orthogonal basis for V as claimed. Normalizing each w_i will then yield an orthonormal basis for V.

The above construction is known as the *Gram–Schmidt orthogonalization process*. The following remarks are in order.

Remark 1: Each vector w_k is a linear combination of v_k and the preceding w's. Hence one can easily show, by induction, that each w_k is a linear combination of $v_1, v_2, \ldots, v_n$.

Remark 2: Since taking multiples of vectors does not affect orthgonality, it may be simpler in hand calculations to clear fractions in any new w_k, by multiplying w_k by an appropriate scalar, before obtaining the next w_{k+1}.

Remark 3: Suppose $u_1, u_2, \ldots, u_r$ are linearly independent, and so they form a basis for $U = \text{span}(u_i)$. Applying the Gram–Schmidt orthogonalization process to the u's yields an orthogonal basis for U.

The following theorem (proved in Problems 7.26 and 7.27) use the above algorithm and remarks.

Theorem 7.9: Let $\{v_1, v_2, \ldots, v_n\}$ by any basis of an inner product space V. Then there exists an orthonormal basis $\{u_1, u_2, \ldots, u_n\}$ of V such that the change-of-basis matrix from $\{v_i\}$ to $\{u_i\}$ is triangular, that is, for $k = 1, \ldots, n$,

$$u_k = a_{k1}v_1 + a_{k2}v_2 + \ldots + a_{kk}v_k$$

Theorem 7.10: Suppose $S = \{w_1, w_2, \ldots, w_r\}$ is an orthogonal basis for a subspace W of a vector space V. Then one may extend S to an orthogonal basis for V, that is, one may find vectors $w_{r+1}, \ldots, w_n$ such that $\{w_1, w_2, \ldots, w_n\}$ is an orthogonal basis for V.

Example 7.10. Apply the Gram–Schmidt orthogonalization process to find an orthogonal basis and then an orthonormal basis for the subspace U of $\mathbf{R}^4$ spanned by

$$v_1 = (1, 1, 1, 1), \qquad v_2 = (1, 2, 4, 5), \qquad v_3 = (1, -3, -4, -2)$$

(1) First set $w_1 = v_1 = (1, 1, 1, 1)$.

(2) Compute

$$v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = v_2 - \frac{12}{4} w_1 = (-2, -1, 1, 2)$$

Set $w_2 = (-2, -1, 1, 2)$.

(3) Compute

$$v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 = v_3 - \frac{(-8)}{4} w_1 - \frac{(-7)}{10} w_2 = \left(\tfrac{8}{5}, -\tfrac{17}{10}, -\tfrac{13}{10}, \tfrac{7}{5}\right)$$

Clear fractions to obtain $w_3 = (-6, -17, -13, 14)$.

Thus w_1, w_2, w_3 form an orthogonal basis for U. Normalize these vectors to obtain an orthonormal basis $\{u_1, u_2, u_3\}$ of U. We have $\|w_1\|^2 = 4$, $\|w_2\|^2 = 10$, $\|w_3\|^2 = 910$, so

$$u_1 = \frac{1}{2}(1, 1, 1, 1), \qquad u_2 = \frac{1}{\sqrt{10}}(-2, -1, 1, 2), \qquad u_3 = \frac{1}{\sqrt{910}}(16, -17, -13, 14)$$

Example 7.11. Let V be the vector space of polynomials $f(t)$ with inner product $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$. Apply the Gram–Schmidt orthogonalization process to $\{1, t^2, t^3\}$ to find an orthogonal basis $\{f_0, f_1, f_2, f_3\}$ with integer coefficients for $\mathbf{P}_3(t)$.

Here we use the fact that, for $r + s = n$,

$$\langle t^r, t^s\rangle = \int_{-1}^{1} t^n\,dt = \left.\frac{t^{n+1}}{n+1}\right|_{-1}^{1} = \begin{cases} 2/(n+1) & \text{when } n \text{ is even} \\ 0 & \text{when } n \text{ is odd} \end{cases}$$

(1) First set $f_0 = 1$.

(2) Compute $t = \dfrac{\langle t, 1\rangle}{\langle 1, 1\rangle}(1) = t - 0 = t$. Set $f_1 = t$.

(3) Compute

$$t^2 - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle}(1) - \frac{\langle t^2, t\rangle}{\langle t, t\rangle}(t) = t^2 - \frac{\frac{2}{3}}{2}(1) + 0(t) = t^2 - \tfrac{1}{3}$$

Multiply by 3 to obtain $f_2 = 3t^2 = 1$.

(4) Compute

$$t^3 - \frac{\langle t^3, 1\rangle}{\langle 1, 1\rangle}(1) - \frac{\langle t^3, t\rangle}{\langle t, t\rangle}(t) - \frac{\langle t^3,\ 3t^2 - 1\rangle}{\langle 3t^2 - 1,\ 3t^2 - 1\rangle}(3t^2 - 1)$$
$$= t^3 - 0(1) - \frac{\frac{2}{5}}{\frac{2}{3}}(t) - 0(3t^2 - 1) = t^3 - \tfrac{3}{5}t$$

Multiply by 5 to obtain $f_3 = 5t^3 - 3t$.

Thus $\{1,\ t,\ 3t^2 - 1,\ 5t^3 - 3t\}$ is the required orthogonal basis.

Remark: Normalizing the polynomials in Example 7.11 so that $p(1) = 1$ yields the polynomials

$$1,\ t,\ \tfrac{1}{2}(3t^2 - 1),\ \tfrac{1}{2}(5t^3 - 3t)$$

These are the first four *Legendre polynomials*, which appear in the study of differential equations.

7.8 ORTHOGONAL AND POSITIVE DEFINITE MATRICES

This section discusses two types of matrices that are closely related to real inner product spaces V. Here vectors in $\mathbf{R}^n$ will be represented by column vectors. Thus $\langle u, v\rangle = u^T v$ denotes the inner product in Euclidean space $\mathbf{R}^n$.