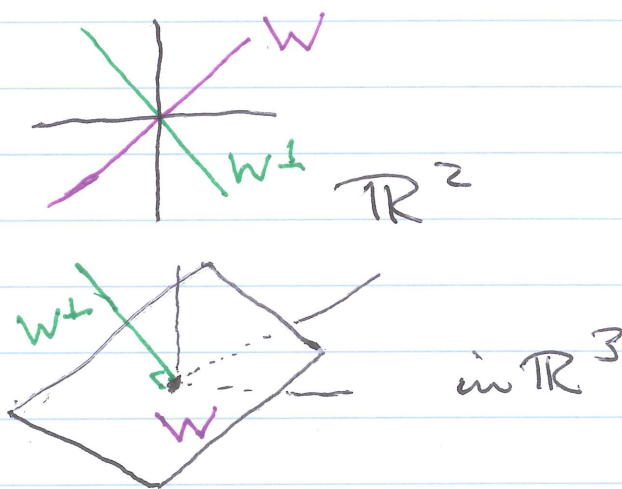


6.C Orthogonal complements and least squares

$V =$ inner product space, $W \subseteq V$ subspace.
The orthogonal complement $W^\perp =$ all $x \in V$ such
that $\langle x, w \rangle = 0$ for all $w \in W$.



Simple facts.

W^\perp is a subspace

$$x, y \in W^\perp \quad c \text{ scalar} \Rightarrow$$

$$\langle x+y, w \rangle = \langle x, w \rangle + \langle y, w \rangle = 0 + 0 = 0$$

all $w \in W$

$$\langle cx, w \rangle = c \langle x, w \rangle = c \cdot 0 = 0 \text{ all } w.$$

$$\{0\}^\perp = V, \quad V^\perp = \{0\}, \quad W \cap W^\perp = \{0\} \text{ all } W$$

$$W_1 \subseteq W_2 \Rightarrow W_2^\perp \subseteq W_1^\perp.$$

Proof. $W + W^\perp = V, W \cap W^\perp = \{0\} \implies$

$$\begin{aligned} \dim W + \dim W^\perp &= \dim (W + W^\perp) + \dim (W \cap W^\perp) \\ &= \dim V + 0. \blacksquare \end{aligned}$$

Corollary $\dim V$ finite \implies

$$W = (W^\perp)^\perp.$$

Proof. $\dim W^{\perp\perp} = \dim W - \dim W^\perp = \dim W.$

On the other hand ~~$x \in W^{\perp\perp}$~~

$$x \in W \implies \langle x, y \rangle = 0 \text{ all } y \in W^\perp \implies \overset{\text{x belongs to}}{\uparrow} W^{\perp\perp}.$$

Hence $W \subseteq W^{\perp\perp}$. If $r = \dim W = \dim W^{\perp\perp}$,

then if $\{w_1, \dots, w_r\}$ is a basis for W , it must also be a basis for $W^{\perp\perp}$, which has the same

dimension. Hence $W \supseteq W^{\perp\perp}$, so $W = W^{\perp\perp}$. \blacksquare

Orthogonal Projection onto W

$V =$ finite dimensional inner product space,
 $W \subseteq V$ subspace. The orthogonal (perpendicular)
 projection $E_W(v)$ of v onto W is defined by
 x_1 , where $v = x_1 + x_2$ where $x_1 \in W, x_2 \in W^\perp$.

As before, if $\{w_1, \dots, w_r\}$ is an orthonormal basis for W , then $E_W(x) = \sum \langle x, w_j \rangle w_j$.

Prop (^{obscure in} ~~missing from Axler!~~) E_W is linear.

Verification $E_W(x+y) = \sum \langle x+y, w_j \rangle w_j = \dots$

$$\sum \langle x, w_j \rangle w_j + \langle y, w_j \rangle w_j = \dots = E_W(x) + E_W(y).$$

$$E_W(cx) = \sum \langle cx, w_j \rangle w_j = \sum c \langle x, w_j \rangle w_j =$$

$$c \sum \langle x, w_j \rangle w_j = c E_W(x). \blacksquare$$

Further properties

(i) $E_W(x) = x$ if $x \in W$, $E_W(x) = 0$ if $x \in W^\perp$.

(ii) $(E_W)^2 x = E_W x$.

Verification of (ii) Write $x = x_1 + x_2$ as before. Then $E^2 x = E(Ex) = E(E(x_1 + x_2)) =$

$EEx_1 = Ex_1 = x_1$. But $x_1 = Ex$. Since

$E^2 x = Ex$ for all x , we have $E^2 = E$. \blacksquare

Example Let $W = \text{Span of } (2, 3, 4)$
and $(0, 3, 4)$. If $x = (1, 1, 0)$, find
 $E_W(x)$.

Sketch of computation First find orthonormal
basis for W . Gram-Schmidt yields
 $(0, \frac{3}{5}, \frac{4}{5})$ & $(1, 0, 0)$. Therefore $E_W(1, 1, 0) =$
 $\langle (0, \frac{3}{5}, \frac{4}{5}), (1, 1, 0) \rangle (0, \frac{3}{5}, \frac{4}{5}) + \langle (1, 0, 0), (1, 1, 0) \rangle (1, 0, 0) =$
 $\frac{3}{5} \cdot (0, \frac{3}{5}, \frac{4}{5}) + 1 \cdot (1, 0, 0) = (1, \frac{9}{25}, \frac{12}{25})$.

Least squares principle

$W \subseteq V$ fin dim inner product space.

For $x \in V$, find $y \in W$ so that $|x - y|^2$ is as
small as possible.

Solution The minimum occurs when $y = E_W x$
and nowhere else.

Verification ~~Write $x = y_1 + y_2$ where $y_1 \in$~~

Verification Write $x = x_1 + x_2$ where $x_1 \in W, x_2 \in W^\perp$. Then $\|x - y\|^2 =$

$\|x_2 + (x_1 - y)\|^2$, and since $x_2 \perp x_1 - y$ we know that \uparrow equals $\|x_2\|^2 + \|x_1 - y\|^2 \geq \|x_2\|^2$.

Equality holds $\iff x_1 - y = 0$ or $y = x_1 = E_W x$. \square

Cor. If w_1, \dots, w_r is an orthonormal basis for W , then $\|x - \sum_{j=1}^r c_j w_j\|^2$ is minimized \iff

$c_j = \langle x, w_j \rangle$ all j . \square

Fundamental Application: Want an

empirical formula for one variable y as $\sum_{j=1}^n a_j x_j + b$ in terms of variables x_1, \dots, x_n .

We may try to do this by measuring x_1, \dots, x_n, y in m cases, where m is much larger than $n+1$. Usually it is not possible to find an exact solution for a_1, \dots, a_n, b . The best we can hope for is a "least squares soln."

Suppose we are given column vectors

$$X_1, \dots, X_n \quad X_j = \begin{pmatrix} x_{1j} \\ \vdots \\ x_{mj} \end{pmatrix} \text{ of } m \text{ observations}$$

in which the i th observation yields x_{ij} for variable x_j

and similarly for Y . Let $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = U$. Find

constants a_1, \dots, a_n, b so that

$$\left| Y - \left(\sum a_j X_j + b U \right) \right|^2 \text{ is minimized.}$$

If $\{X_1, \dots, X_n, U\}$ is linearly independent the methods of this section and the previous one can be used to find the coefficients a_1, \dots, a_n, b . However, in most cases it is much better to use more systematic methods that would require too much time to describe (but are not much more complicated!).