7A. Linear transformations and inner products

DEFAULT HYPOTHESIS. All vector spaces are finite dimensional.

Theorem ("Riesz Representation" in Axler; actually well known much earlier in the 19th century.

Notation If V is a vector space, a linear functional is a linear transformation $V \longrightarrow \mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$).

If V is equipped with an inner product $\langle \, , \rangle$ and $f : V \longrightarrow \mathbb{F}$ is a linear functional, then there is a unique $x_f \in V$ such that for all $y \in V$ we have $f(y) = \cancel{\langle x_f, y \rangle}$ $\langle y, x_f \rangle$.

Derivation Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis for V.

Existence. Suppose that $f(u_j) = c_j$. Let $x_f = \sum \overline{c_j} u_j$. note the conjugation. Then $f(u_j) = \langle u_j, x_f \rangle$ all j, and if $y = \sum a_k u_k$, then

$$f(y) = \sum a_k f(u_k) = \sum a_k c_k =$$

$$\langle \sum a_k w_k, \sum \bar{c}_j w_j \rangle = \langle y, x_f \rangle. \blacksquare$$

Uniqueness. Suppose $\langle y, x \rangle = \langle y, x' \rangle$ for all $y \in V$. Then $\langle y, x - x' \rangle = 0$ all y, so $x - x' \in V^{\perp} = \{0\}$. $\blacksquare$

Adjoint transformations

Theorem. Let $T : V \to W$ be a linear transf. of inner product spaces. Then there is a unique linear transformation T^*: ~~$W \to V^*$~~ $W \to V$ such that

$$\langle Tv, w \rangle_W = \langle v, T^* w \rangle_V.$$

This map T^* is called the adjoint transformation.

Proof. Existence Notice that if $w \in W$ then $f(v) = \langle Tv, w \rangle$ is a linear functional on V. Therefore $\langle Tv, w \rangle = \langle v, z \rangle$ for some unique $z \in V$; set $T^* w = v$.

Need to show T^* is linear: $\langle v, T^*(w_1 + w_2) \rangle =$
$\langle Tv, w_1 + w_2 \rangle = \langle Tv, w_1 \rangle + \langle Tv, w_2 \rangle =$

$\langle v, T^* w_1 \rangle + \langle v, T^* w_2 \rangle = \langle v, T^* w_1 + T^* w_2 \rangle$ for all $v \in V$. By the previous result, $T^*(w_1 + w_2) = T^* w_1 + T^* w_2$. Also

$$\langle v, T^* cw \rangle = \langle Tv, cw \rangle = \bar{c} \langle Tv, w \rangle = \bar{c} \langle v, T^* w \rangle = \langle v, \bar{c} T^* w \rangle,$$

and as before $T^*(cw) = \bar{c} T^* w$. ■

Formal properties $(S+T)^* = S^* + T^*$

$(cT)^* = \bar{c} T^*$

$T^{**} = T \qquad I^* = I$

$(ST)^* = T^* S^*$.

Derivations are on Axler, p. 206

Matrix representation Let $\{v_1, \dots, v_n\}$ V

and $\{w_1, \dots, w_m\}$ be orthonormal bases for
W

V & W respectively, and let $T: V \to W$ be linear.

CLAIM If $T v_j = \sum_i a_{ij} w_i$, then $\langle T v_j, w_i \rangle = a_{ij}$.

Then $T^* w_j = \sum b_{ij} v_i \Longrightarrow$

$$b_{ij} = \langle T^* w_j, v_i \rangle = \overline{\langle v_i, T^* w_j \rangle} = \overline{\langle T v_i, w_j \rangle} = \overline{a_{ji}}$$

conjugate transpose.

(ordinary transpose over $\mathbb{R}$)

More identities

$$\text{Kernel } T^* = (\text{Image } T)^\perp$$
$$\text{Image } T^* = (\text{Kernel } T)^\perp$$
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$$\text{Image } T = (\text{Kernel } T^*)^\perp.$$

Note that the first two and $T^{**} = T$ imply the last two.

See Axler, p. 207, for derivations.

Adjoints and diagonalization

Goal: Show that if T and $T^*: V \to V$ are closely related (for example, $T = T^*$), then T has an orthonormal basis of eigenvectors.

Def. $T: V \to V$ is self-adjoint if $T = T^*$.

In terms of matrices, if T is rep. by A, then T^* is rep by the $\boxed{\text{conjugate transpose}}$ $\overline{{}^TA} = A^*$. (${}^TA = A$ transposed).

If A is a matrix with real entries, $A^* = A$ means ${}^TA = A$, or A is symmetric.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{typical example.}$$

Over $\mathbb{C}$, we say A is Hermitian if $A^* = A$
(her-MEE-shen)
(after C. Hermite, 1822-1901)

typical example $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

PROPOSITION. If $T^* = T$, then all eigenvalues are real.

Proof Say $Tv = cv$, $v \neq 0$. Then

$$c \cdot |v|^2 = c\langle v, v\rangle = \langle cv, v\rangle = \langle Tv, v\rangle$$

$$= \langle v, T^* v\rangle = \langle v, cv\rangle = \bar{c}\langle v, v\rangle = \bar{c}|v|^2.$$

Since $|v|^2 > 0$, we can cancel it and conclude that $c = \bar{c}$, so that c is real. ∎

THEOREM. (Principal Axis Theorem) If A is real symmetric, then A has a real eigenvalue.

Proof If A is invertible over $\mathbb{R}$, then A is invertible over $\mathbb{C}$. Taking contrapositives, if A is not invertible over $\mathbb{C}$ and A is real, then A is not invertible over $\mathbb{R}$.

${}^{T}A = A$ and the previous result imply $A - cI$ is not $\mathbb{C}$ invertible for some $c \in \mathbb{R}$, so it also isn't $\mathbb{R}$ invertible.

Therefore A must have a real eigenvector for $c \in \mathbb{R}$.

What follows is out of order from Axler.

<u>Theorem (Diagonalization)</u> If $A^* = A$ over $\mathbb{R}$ or $\mathbb{C}$, then A has an orthonormal basis of eigenvectors over $\mathbb{R}$ or $\mathbb{C}$.

suffices to find an orthogonal basis.

In particular, real-symmetric matrices are diagonalizable.

<u>Proof</u> Assume $T^* = T$ throughout this proof, which works over $\mathbb{R}$ or $\mathbb{C}$. Prove the result for $T : V \to V$ (inner product space) with $T^* = T$.

<u>$\dim V = 1$</u> True since $0 \neq v \Rightarrow Tv = cv$.

Assume if $\dim V = k-1$, where $k > 1$.

<u>Inductive step</u> Know $Tv = cv$ some $v \neq 0$, scalar c. Claim: If $W = \operatorname{Span}(\{v\})^\perp$, then

$T[W] \subseteq W$ (and $T^*[W] \subseteq W$).

Since $T = T^*$, we know $T^* v = cv$ also holds. Let $x \in W$, so that $\langle v, x \rangle = 0$. But then $\langle v, Tx \rangle = \langle Tv, x \rangle = \langle cv, x \rangle = c \langle v, x \rangle = c \cdot 0 = 0$. If we let $S: W \to W$ be the associated linear transformation, then $S^* = S$, so by induction S has an orthonormal/gonal basis of eigenvectors $v_2, \ldots, v_k$. If $v = v_1$, then $\{v_1, \ldots, v_k\}$ yields an orthogonal basis of eigenvectors for T. ∎

We shall discuss some fundamentally important mathematical consequences of this theorem (over $\mathbb{R}$) at the end of this chapter (probably in lieu of Section 7D in Axler).