7B. Normal operators and the Spectral Theorem

<u>Def.</u> A linear transformation $T: V \to V$ (operator) on an inner product space is <u>normal</u> if $TT^* = T^*T$.

As before, assume $\dim V < \infty$.

<u>Examples</u> Self adjoint transformations $T = T^*$

Skew adjoint transformations $T^* = -T$

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix} \quad \text{over } \mathbb{R} \ \left(\begin{matrix}\text{skew} \\ \text{symmetric}\end{matrix}\right)$$

<u>ORTHOGONAL MATRICES</u> 2×2 case

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

rotation through θ

orthonormal basis eigenvectors with eigenvalues ± 1. (compare $\theta = 0$).

get $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

In these examples $A \cdot {}^{T}A = {}^{T}A \cdot A = I$.

Complex analog: UNITARY MATRICES

$$AA^* = I = A^*A$$

In both these cases, the underlying principle is that the columns of A form an orthonormal set.

Axler Example, p. 212

$$A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$$

Prop. $T: V \to V$ is normal $\iff$ $|Tv| = |T^*v|$ all v.

Cor. T normal $\Rightarrow$ $\operatorname{Ker} T = \operatorname{Ker} T^*$.

Proof of Prop. T normal $\iff T^*T - TT^* = 0$

$\iff \langle (T^*T - TT^*)v, v \rangle = 0$ all v

(since $T^*T - TT^*$ is self adjoint) $\iff$

$\langle T^*Tv, v \rangle = \langle TT^*v, v \rangle$ all v $\iff$

$\langle Tv, Tv \rangle = \langle T^*v, T^*v \rangle$ all v. ∎

<u>Cor.</u> If v is an eigenvector for T and T is normal, then v is also an eigenvector for T^*. Furthermore, if $Tv = cv$, then $T^*v = \bar{c}v$.

<u>Proof.</u> v is an eigen~~value~~vector of T with eigenvalue c
$\iff v \in \operatorname{Ker}(T - cI) \iff v \in \operatorname{Ker}(T^* - \bar{c}I)$
(since $T - cI$ is normal and $(T-cI)^* = T^* - \bar{c}I$)
$\iff v$ is an eigenvector of T^* with eigenvalue $\bar{c}$. ∎

<u>Prop.</u> If T is normal, then eigenvectors for distinct eigenvalues are orthogonal.

<u>Proof.</u> Say $Tw = aw$, $Tv = bv$, where $v, w \neq 0$ and $a \neq b$. Then

$$\langle Tw, v\rangle = \langle aw, v\rangle = a\langle w, v\rangle$$
$$\|$$
$$\langle w, T^*v\rangle = \langle w, \bar{b}v\rangle = b\langle w, v\rangle .$$

So $a\langle w,v\rangle = b\langle w,v\rangle$ with $a \neq b$.

This can only happen if $\langle w,v\rangle = 0$. ∎

spectrum

SPECTRAL THEOREM Over $\mathbb{C}$.

V as usual, $T: V \to V$ linear. Then T has an orthonormal basis of eigenvectors $\Longleftrightarrow$ T is normal.

IMPORTANT. This fails over $\mathbb{R}$. Say A is a 2×2 rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ where $\theta \neq n\pi$ (with $n \in \mathbb{Z}$ integers).

Then $A \cdot {}^TA = {}^TA \cdot A = I$ but there are no real eigenvectors.

$\boxed{\Longleftarrow}$

Proof. Induction on $\dim V$.

$\dim V = 1 \Rightarrow Tv = cv$ all $v \in V$, some c.

Suppose true if $\dim V \leq n-1$ where $n \geq 2$.

Inductive step. We know that T has an eigenvalue c for some eigen~~value~~ vector. Let $W =$ all $x \in V$ so that $Tx = cx$. Then $T^*x = \bar{c}x$ also holds, and W is both

T and T^* invariant.

CLAIM: $W^\perp$ is also invariant under T and T^*.

Proof of claim Suppose $y \in W^\perp$. Then

$T y \in W^\perp \Leftrightarrow \langle x, Ty \rangle = 0$ all $x \in W \Leftrightarrow$

$\langle T^* x, y \rangle = 0$ all x. Since $x \in W \Rightarrow T^* x = \bar{c} x$,

the latter is equivalent to $\bar{c} \langle x, y \rangle = 0$ all $x \in W$.

But the latter holds since $x \in W$ & $y \in W^\perp$.

If we ~~reve~~ switch the roles of T & T^*, $\bar{c}$ & c, we also get $T^* y \in W^\perp$.

Completion of the inductive step. The claim shows that T determines a linear transformation $T_0 : W^\perp \to W^\perp$ and likewise for T^*. These cut down maps are adjoint, and the cut down map T_0 is normal because of this and $TT^* = T^* T$.

Hence $W^\perp$ has an orthonormal basis of eigen vectors by the induction hypothesis.

Also, W has an orthonormal basis of eigenvectors since $Tx = cx$ for all $x \in W$. Combine these bases for W and $W^{\perp}$ to obtain an orthonormal basis ~~for~~ of eigenvectors for the linear transformation T. ■

$\boxed{(\Longrightarrow)}$ Let $\{u_1, \ldots, u_m\}$ be an orthonormal basis with $Tu_j = c_j u_j$.

(CLAIM) $T^* u_k = \bar{c}_k u_k$.

Need to show $\langle v, T^* u_k \rangle = \langle v, \bar{c}_k u_k \rangle$ all v. But LHS $= \langle Tv, u_k \rangle$. Write $v = \sum a_j u_j$, so the latter inner product becomes

$$\left\langle T \sum a_j u_j, u_k \right\rangle = \sum a_j c_j \cancel{\langle u_j, u_k \rangle} =$$

0 if $j \neq k$
1 if $j = k$

~~$\sum a_j \langle u_j,$~~

$a_k c_k$. On the other hand, we also

have $\langle \sum a_j w_j, \bar{c}_k w_k \rangle = a_k c_k$. ∎

Back to the main proof. Thanks to the claim we may write

$$\left.\begin{aligned} T^*Tv &= \sum |c_j|^2 a_j w_j \\ TT^*v &= \sum |c_j|^2 a_j w_j \end{aligned}\right\} \text{ if } v = \sum a_j w_j.$$

Hence $T^*T = TT^*$. ∎