7C. Positive matrices

(We already discussed isometries) orthogonal/unitary linear transf.

There are really two different concepts to consider; all matrices are real.

positive definite ${}^TA = A$ and $\langle Ax, x\rangle > 0$ if $x \neq 0$. (Example: $A = I$)

positive semidefinite ${}^TA = A$ and $\langle Ax, x\rangle \geq 0$ all x. (Axler's notion of positive matrix)

Thm. Suppose ${}^TA = A$. Then A is $\left\{\begin{matrix}\text{positive definite}\\ \text{positive semidefinite}\end{matrix}\right\}$ if and only if all its eigenvalues are $\left\{\begin{matrix}\text{positive}\\ \text{nonnegative}\end{matrix}\right\}$.

Proof ① ($\Rightarrow$) Suppose A is positive definite. If c is an eigenvalue, so $Ax = cx$ for some $x \neq 0$, then $c\langle x, x\rangle = \langle cx, x\rangle = \langle Ax, x\rangle > 0$, so $|x|^2 > 0 \Rightarrow c > 0$.

(⇐) Suppose all eigenvalues are positive and write $x = \sum a_j v_j$. Then $\langle Ax, x\rangle = \sum c_j |a_j|^2$ and if $c^* > 0$ is the minimum eigenvalue, this is $\geq c^* \sum a_j^2 = c^* |x|^2$, which is positive. ■

② (⇒) Similar to ①, but now $\langle Ax, x\rangle \geq 0$ so $|x|^2 > 0 \Rightarrow c \geq 0$. (⇐) Again similar to the preceding except that $c^* \geq 0$ so $c^* |x|^2$ is nonnegative. ■

<u>One reason for interest in positive definite matrices</u> These are precisely the symm. matrices such that $\langle Ax, x\rangle$ is an inner product on $\mathbb{R}^n$.

<u>EXAMPLES.</u> Gram matrices

If P is an invertible $n \times n$ matrix and $G = {}^TPP$ then g_{ij} is the dot product of the columns $\langle P_i, P_j\rangle$.

Conversely,

<u>Theorem</u> If A is positive definite, then there is an invertible matrix Q such that
$$A = {}^TQQ.$$

<u>Proof.</u> Let P be an invertible $n \times n$ matrix, and let p_k be its kth column. Then
$$\langle Ap_j, p_i \rangle \text{ is the } i,j \text{ entry of } {}^TPAP.$$

Since ${}^TYAX = \langle AX, Y \rangle$ defines an inner product, it has an orthonormal basis. Therefore one can find an orthonormal basis; if P is a matrix whose columns are the vectors in such a basis, we then have
$${}^TPAP = I.$$

Let $Q = P^{-1}$. Then ${}^TQ = ({}^TP)^{-1}$, and therefore $A = ({}^TP)^{-1}P^{-1} = {}^TQQ.$ ■

Proof that ${}^T(B^{-1}) = ({}^TB)^{-1}$: $BC = CB = I \Rightarrow$

${}^T(BC) = C^TB^T$, ${}^T(CB) = {}^TB\,{}^TC$, so $I = {}^TI = {}^T(BC) = {}^TC\,{}^TB$ and $I = {}^TI = {}^T(CB) = {}^TB\,{}^TC$. ■

Recognizing 2×2 Pos. Def. Matrices.

$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ real symmetric.

A is positive definite $\Leftrightarrow$ eigenvalues are both positive.

<u>Note</u>. Only one eigenvalue $c \Rightarrow$ $A = cI$, an easy situation to handle.

The eigenvalues are the roots of $0 = \begin{vmatrix} a-t & b \\ b & d-t \end{vmatrix} =$

$t^2 - (a+d)t + (ad - b^2)$.

If λ_1 and λ_2 are the eigenvalues, then

$$\lambda_1 + \lambda_2 = a+d$$
$$\lambda_1 \lambda_2 = ad - b^2.$$

I. If A is pos. def., then $ad - bc > 0$
$a > 0$.

<u>Proof.</u> A pos. def. $\Rightarrow \lambda_1, \lambda_2 > 0 \Rightarrow \lambda_1 \lambda_2 > 0$
$\Rightarrow \boxed{ad - b^2 > 0}$. The latter $\Rightarrow ad > 0$ so $a+d$
have the same sign. But also $\lambda_1 + \lambda_2 > 0$, so

$a+d>0$. Now a, d have the same sign, so $a+d>0$ means this sign is $+$. Hence $a>0$.

II Conversely, $\begin{vmatrix} a & b \\ b & d \end{vmatrix} > 0$ and $a>0$ imply $\lambda_1, \lambda_2 > 0$. — First, $\begin{vmatrix} a & b \\ b & d \end{vmatrix} = \lambda_1 \lambda_2$ positive means λ_1 & λ_2 have the same sign, and likewise for a & d. Now

$a>0 \Rightarrow \lambda_1 + \lambda_2 = a+d > 0$, and this means the common sign for λ_1, λ_2 is $+$. ∎

Examples $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ $\quad 2>0$, $\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5 > 0$

so the matrix is positive definite.

$$\begin{vmatrix} 2-t & 1 \\ 1 & 3-t \end{vmatrix} = t^2 - 5t + 5 \Rightarrow \text{eigenvalues are}$$

$$\frac{5 \pm \sqrt{5}}{2}.$$

$\begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$ $\quad \begin{vmatrix} 5-t & 2 \\ 2 & 2-t \end{vmatrix} = t^2 - 7t + 6 = (t-6)(t-1)$

$\begin{vmatrix} 5 & 2 \\ 2 & 2 \end{vmatrix} = 10$, $5>0$ $\quad A - I = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow v_1 = (1, -2)$

so pos. def $\quad v_2 = (2, 1)$

can read off from perpendicularity