Invertibility criterion for triangular matrices

Recall that a square matrix $A = (a_{i,j})$ is *upper triangular* if all entries below the main diagonal are zero; in terms of equations, this means $a_{i,j} = 0$ if $i > j$. The goal is to prove the following assertion:

THEOREM. *Let $A = (a_{i,j})$ be a square upper triangular matrix. Then A is invertible if and only if $a_{j,j} \neq 0$ for all j (in other words, all diagonal elements are nonzero).*

We shall prove that if all diagonal elements are nonzero, then A is invertible, and conversely if some diagonal element is zero then A is not invertible.

Proof. Let n be the number of rows and columns in A, and let α_j denote the j^{th} column of A. Also, let $\mathbf{e}_j$ denote the j^{th} column unit vector with 1 in position j and zeros elsewhere.

Suppose first that all diagonal entries are nonzero. We shall prove by induction that $\mathbf{e}_j$ is a linear combination of $\alpha_1, \cdots, \alpha_j$. If we then take $j = n$, then it will follow that $\alpha_1, \cdots, \alpha_n$ span the space of $n \times 1$ column vectors and hence form a basis. Since a square matrix is invertible if and only if its columns form a basis, it follows that A will be invertible.

If $j = 1$ then the inductive assertion is trivially true because $\alpha_1 = a_{1,1}\,\mathbf{e}_1$ and hence $\mathbf{e}_1 = a_{1,1}^{-1}\,\alpha_1$. — Suppose now that the assertion is known for $k \leq j-1$, where $j - 1 \geq 1$. By construction we have $\alpha_j = a_{j,j}\,\mathbf{e}_j + \beta$ where β is a linear combination of $\mathbf{e}_1, \cdots, \mathbf{e}_{j-1}$. Then we have

$$\mathbf{e} \;=\; a_{j,j}^{-1}\,\alpha_j \;-\; \beta$$

and by the defining formula and inductive hypothesis we know that β is a linear combination of $\alpha_1, \cdots, \alpha_{j-1}$. This establishes the inductive hypothesis for $j \leq k$ where $k \leq n$, and as noted above it implies that A is invertible.■

Conversely, suppose that some diagonal entry is zero. Specifically, choose k to be the first integer such that $a_{k,k} = 0$. Then by the reasoning of the first part we know that the unit vectors $\mathbf{e}_1, \cdots, \mathbf{e}_{k-1}$ are linear combinations of the vectors $\alpha_1, \cdots, \alpha_{k-1}$. Since α_k is a linear combination of $\mathbf{e}_1, \cdots, \mathbf{e}_{k-1}$ by the condition $a_{k,k} = 0$, it follows that α_k is also a linear combination of $\alpha_1, \cdots, \alpha_{k-1}$ if $k - 1 \geq 1$; assume this temporarily — we shall need to dispose of the case $k = 1$ after we are finished with the case $k > 1$. Therefore the columns of A are linearly dependent and hence do not form a basis for the space of column matrices.

All that remains is to consider the case $k = 1$. However, in this case $\alpha_k = 0$, and since a set of vectors is linearly dependent if it contains the zero vector, in this case we also see that the columns of A are linearly dependent and hence do not form a basis for the space of column matrices.■

COROLLARY. *If A is a square upper triangular matrix, then the eigenvalues of A are the triangular matrices.*

Proof. We know that c is an eigenvalue of A if and only if $A - cI$ is not invertible, and since A is upper triangular this holds if and only if some diagonal entry of $A - cI$ is zero; note that the latter matrix is also upper triangular. Since the condition on diagonal entries for $A - cI$ is $a_{j,j} - c = 0$ for some j, it follows immediately that c is an eigenvalue if and only if $a_{j,j} = c$ for some j.■