Cayley – Hamilton Theorem for Jordan form matrices

If A is an $n \times n$ matrix, the Cayley – Hamilton Theorem describes an explict polynomial $\chi_A(t)$ of degree n (the characteristic polynomial) such that $\chi_A(A) = 0$. One can define it using determinants, but since Axler does not treat these until the last chapter in the book we have to give an *ad hoc* description here which is valid if A is in Jordan form. Namely, if A is in Jordan form (hence A is upper triangular), then

$$\chi_A(t) \quad = \quad \prod_{j=1}^{n} (t - a_{j,j}) \ .$$

Derivation of the Cayley – Hamilton Theorem for Jordan form matrices. A matrix in Jordan form is a block sum of elementary Jordan $k \times k$ matrices $(c_{i,j})$ such that $c_{i,i} = \lambda$ for some fixed λ and all values of $i = 1$ through k, and $c_{i,j+1} = 1$ for $i = 1, ..., k-1$ with $c_{i,j} = 0$ in all remaining cases. If C is such a matrix, then it follows immediately that $\chi_C(C) = (C - \lambda I)^n = 0$.

Suppose now that A is a block sum

$$A \quad = \quad \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$$

where P and Q are square matrices. For upper triangular matrices of this type we have $\chi_A = \chi_P \cdot \chi_Q$, and it follows that if $\chi_P(P) = 0$ and $\chi_Q(Q) = 0$, then

$$\chi_A(A) \quad = \quad \chi_P(A) \cdot \chi_Q(A) \quad = \quad \begin{pmatrix} \chi_P(P)\chi_Q(P) & 0 \\ 0 & \chi_P(Q)\chi_Q(Q) \end{pmatrix}$$

is zero. By induction a similar result holds for block sums with an arbitrary finite number of summands, and if A is in Jordan form, say $A \sim B_1 \oplus \cdots \oplus B_r$, where each B_j is an elementary Jordan matrix, then we see that

$$\chi_A(t) \quad = \quad \prod \chi_{B_j}(t) \qquad \text{and} \qquad \chi_A(A) \quad = \quad \prod \chi_{B_j}(B_j)$$

and by the previous discussion each factor on the right hand side of the second equation is zero. This yields the desired identity $\chi_A(A) = 0$.■

Final remark. The characteristic polynomial can be defined for an arbitrary square matrix, and it has the following key property: If $B = P^{-1}AP$ where P is an invertible matrix, then $\chi_B = \chi_A$. However, the definition of the polynomial and the proof of its key property use the theory of determinants, which has not yet been covered in the course.