8B. Rational decomposition of nilpotent transformations

Review of 8A

Let $T: V \to V$ be a linear transformation, where V is a finite dimensional complex vector space. Then V has a direct sum decomposition

$$\bigoplus_{\lambda} V_{\lambda} \quad \text{(where } \lambda \text{ runs through the eigenvalues)}$$

such that T maps each V_λ into itself and on V_λ we have $(T-\lambda I)^{n(\lambda)} = 0$ for some positive integer $n(\lambda)$.

The preceding allows us to find a "block diagonal" matrix representing T. Let $\mathcal{B}_\lambda$ be an ordered basis for V_λ and order the eigenvalues as $\lambda_1, \ldots, \lambda_k$. Then on each V_{λ_j} we have $T = \lambda_j I + N_j$ where $N_j^{n(\lambda_j)} = 0$. We can then assemble these into the following matrix for T:

$$
\begin{array}{c|cccc}
 & B_{\lambda_1} & B_{\lambda_2} & \cdots & B_{\lambda_k} \\
\hline
B_{\lambda_1} & & & & \\
B_{\lambda_2} & & & & \\
\vdots & & & & \\
B_{\lambda_k} & & & &
\end{array}
\quad
\begin{pmatrix}
\lambda_1 I + B_1 & 0 & \cdots & 0 \\
0 & \lambda_2 I + B_2 & & 0 \\
\vdots & & \ddots & \vdots \\
0 & \cdots & & \lambda_k I + B_k
\end{pmatrix}
$$

Each of the matrices B_j is nilpotent.*

To complete the picture, we want matrices of the B_j for the subspaces V_{λ_j} which have as many zero entries as possible. We can do this individually for each j.

Historical note The term "nilpotent" (accent on the second syllable), like much of linear algebra, is due to J. J. Sylvester (1814-1897). Biographical information about Sylvester is available on line, either through Wikipedia or the University of St. Andrews history of mathematics site.

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*B is nilpotent if $B^q = 0$ for some q. The least q is called the index of nilpotency

www-groups.dcs.st-and.ac.uk/history/biography/

~~James Joseph Sylvester~~
Sylvester.html

Iterations of linear transformations

Since $T^m v = 0 \Longrightarrow T^{m+1} v = 0$

($T: V \to V$ a linear transformation)

we have

$$\text{Kernel } T \subseteq \text{Kernel } T^2 \subseteq \dots\dots \subseteq V.$$

Suppose that $\dim V = n$, and let $d_j = \dim (\text{Ker } T^j)$. Then we have

$d_1 \leq d_2 \leq \dots \leq n = \dim V$. Since the d_j's are integers, this sequence is eventually constant.

In fact, if two consecutive terms are equal, then all ~~remain~~ further terms take the same value.

<u>Lemma 1</u> If $\operatorname{Ker} T^m = \operatorname{Ker} T^{m+1}$, then $\operatorname{Ker} T^{m+j} = \operatorname{Ker} T^m$ for all $j \geq 0$.

<u>PROOF.</u> Will show the conclusion by induction on j. The result is known if $j = 0$ or 1. If it is true for $j = p$, then we need to show $\operatorname{Ker} T^{p+m} = \operatorname{Ker} T^{p+m+1}$. We know $(\subseteq)$ is true. To check $(\supseteq)$, suppose $T^{p+m+1} x = 0$. Then $T^p x \in \operatorname{Ker} T^{m+1}$. But the latter is $\operatorname{Ker} T^m$, and hence $T^m T^p x = 0$, so that $\operatorname{Ker} T^{p+m+1} = \operatorname{Ker} T^{p+m}$, and we know the latter is just $\operatorname{Ker} T^m$. ∎

Specialize to the nilpotent case.

<u>Lemma 2</u> (i) If T is nilpotent, then $T^n = 0$.

(ii) If T is nilpotent and $k > 0$ is minimal so that $T^k = 0$, then $0 < d_1 < \ldots < d_k = n$.

PROOF. The second conclusion follows because of Lemma 1. To see the second note that if $0 \leq d_1 < \dots < d_k = n$, then $k \leq n$, so it suffices to prove that $d_1 > 0$; i.e., $\operatorname{Ker} T \neq 0$. However, if $\operatorname{Ker} T \neq 0$ then T is invertible and each power of T has $\operatorname{Ker} T^j = 0$. This is impossible if T is nilpotent, for then $\operatorname{Ker} T^s = V$ for some large s. ∎

Corollary 3 T nilpotent $\Rightarrow$ T is not invertible. ∎

Let's analyze the special case of a nilpotent $N : V \to V$ such that $N^2 = 0$.

Note first that $\text{Image } N \subseteq \text{Kernel } N$ in such cases. Let $x_1, \dots, x_p$ be a basis for $\operatorname{Im}(N)$, and expand it to a basis for $\operatorname{Ker}(N)$ by adjoining $y_1, \dots, y_q$. Finally let $z_1, \dots, z_p$ be such that $Tz_i = x_i$. Then we can represent N by the following matrix:

$\cancel{z_1 \; x_1 \; z_2 \; x_2 \cdots z_p \; x_p}$

$$
\begin{array}{c|cc|cc|cc|c}
 & x_1 & z_1 & x_2 \;\; z_2 & \cdots & x_p & z_p & y_1 \cdots y_q \\
\hline
x_1 & 0 & 1 & & & & & \\
z_1 & 0 & 0 & & & & & 0 \\
\hline
x_2 & & & 0 \;\; 1 & & & & \\
z_2 & 0 & & 0 \;\; 0 & & & & 0 \\
\hline
\vdots & 0 & & & \ddots & & & 0 \\
\hline
x_p & & & & & 0 & 1 & \\
z_p & 0 & & 0 & & 0 & 0 & 0 \\
\hline
y_1 & & & & & & & \\
\vdots & & & & & & & \\
y_q & 0 & & 0 & & 0 & & 0
\end{array}
$$

This is a minimally sparse matrix for a linear transformation with rank p. (At least p columns must have nonzero entries!)

Strategy for the case where $N^3 = 0$. $(N^2 \neq 0)$.

Let $V_0 = \text{Image } N$. Then N sends V_0 to itself. If $N_0 : V_0 \to V_0$ is given by the same formula as N, then $N_0^2 = 0$

$(N^2(Ny) = N^3 y = 0y = 0)$.

Construct a basis for V_0 with respect to N_0:

$$z_1, \dots, z_p, y_1, \dots y_q$$
$$Nz_1, \dots Nz_p$$

Now choose $z_i', y_j' \in V$ such that

$$Nz_i' = z_i, \quad Ny_j' = y_j$$

obtaining a new (linearly independent) set in V:

$$X = \left\{ \begin{array}{ll} z_1', \dots, z_p' & y_1', \dots, y_q' \\ Nz_1', \dots, Nz_p' & Ny_1', \dots, Ny_q' \\ N^2z_1', \dots, N^2z_p' & \end{array} \right\}$$

We need to verify this set is linearly independent.

Suppose that

$$\sum a_j z_j' + \sum b_j N z_j' + \sum c_j N^2 z_j'$$

$$+ \sum d_j y_j + \sum e_j N y_j' = 0$$

If we apply N, we get

$$\sum a_j z_j + \sum b_j N z_j + \sum d_j y_j = 0$$

and by construction the a_j's, b_j's and d_j's are all 0. Substituting this into the top equation, we get

$$\sum c_j N z_j + \sum e_j y_j = 0$$

and again by construction this yields $c_j = 0$, $e_j = 0$ for all j.

<u>NOW</u> expand X to a basis for V by adjoining $\{u_1, \ldots, u_r\}$. The final step is to modify these slightly, obtaining $\{w_1, \ldots, w_r\}$ with $N w_j = 0$ for all j.

Specifically, We know

$$N w_j = N v_j \text{ for some } v_j \in \mathrm{Span}(X)$$

If $w_j = w_j - v_j$ then $X \cup \{w_1, \ldots, w_r\}$ is also a basis for V, but now $N w_j = 0$ all j.

Thus there is a basis for V of the form.

$$\left\{ \begin{array}{ccc} N^i z_j' & N^i y_j' & w_j \\ i = 0, 1, 2 & i = 0, 1 & \\ \hline N^3 z_j' = 0 & N^2 y_j' = 0 & N w_j = 0 \end{array} \right\}. \blacksquare$$

NOTE. Either the second or third column (or both) might be empty!

RECURSIVE REFORMULATION

Induction on the index of nilpotency (= least $p > 0$ such that $N^p = 0$).

Examine the proof when $p = 3$, then generalize the conclusion.

We have a basis $N^i y_{k,j}$ for V each piece of which has the form

$$y_{k,j} = N^0 y_{k,j},\ N^1 y_{k,j},\ \ldots,\ N^{m(k)-1} y_{k,j}$$

such that $N^{m(k)} y_{k,j} = 0$, where $1 \le j \le q(k)$, and $p = m(1) \ge m(2) \ge \ldots \ (\ge 1)$. [$p$ = index of nilp.]

Assume this is known if $N^p = 0$, and now suppose we have N with $N^{p+1} = 0$ but $N^p \ne 0$.

Let $V_0 = \text{Image}\, N$, so that $N[V_0] \subseteq V_0$, and let $N_0 : V_0 \longrightarrow V_0$ be defined by $N_0(x) = Nx$. Then $y \in V_0 \Longrightarrow y = Nx \Rightarrow N^p y = N^{p+1} x = 0$, so that $N_0{}^p = 0$. Also, if $z \in V$ is such that $N^p z \ne 0$, then $N_0{}^{p-1}(Nz) \ne 0$, so the index of nilpotency for N_0 is equal to p.

<u>Now apply the induction hypothesis</u>

Let $\{N^i y_{k,j} \mid i, j, k\}$ be the basis for V_0 as described on the preceding page, and write $y_{k,j} = N x_{k,j}$ (can do since $V_0 = N[V]$). Consider the set with

$$x_{k,j} = N^0 x_{k,j},\ N^1 x_{k,j}, \ldots\ N^{m(k)} x_{k,j}$$

$$N^1 x_{k,j} = y_{k,j}, \qquad N^{m(k)} x_{k,j} = N^{m(k)-1} y_{k,j}$$

As in the proof when $p+1=3$, this set is linearly independent. Now expand it to a basis for V by adjoining $z'_1, \ldots, z'_r$. Then each z'_ℓ has the property $N z'_\ell = w'_\ell$ where the latter is a linear combination of the given basis for V_0, and $w'_\ell = N w_\ell$ where w_ℓ is a linear comb. of the $N^i x_{k,j}$'s. The new set still spans V (hence is also a basis), but now $N(z_\ell) = 0$, all ℓ.

$z_\ell = z'_\ell - w_\ell$

sub-script "ell"

This completes the proof that there is a basis for V of the form appearing on p. 10, and hence completes the proof of the inductive step. ■

Conclusion, stated in terms of matrices

If N is nilpotent, then N can be represented by a block sum of matrices which are zero off the diagonal and have diagonal pieces like the following:

$$\begin{pmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & & \\ & & & & \ddots & \\ & & & & & \ddots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Formally, $a_{ij} = 1$ if $j = i+1$ and 0 otherwise. ■