8C–8D. Jordan Canonical Form

Summary of the preceding two sections:

Let V be a fin dim vsp over $\mathbb{C}$, and let $T: V \to V$ be a linear transformation. Then V splits into a direct sum of subspaces $V_{\lambda_j, k}$ such that ① each is T-invariant,

② $(T - \lambda_j I)$ is nilpotent on $V_{j,k}$ for each j, k,

③ There is a basis for $V_{j,k}$ of the form

$$x, (T-\lambda_j I)x, (T-\lambda_j I)^2 x, \ldots (T-\lambda_j I)^{m(j,k)-1} x$$

where $m(j,k)$ is the index of nilpotency for $(T-\lambda_j I)$ viewed as a linear transformation from $V_{j,k}$ to itself.

If we form a basis for V by taking the union of the bases in 3 for each $V_{\lambda_j,k}$, then the matrix for T with respect to this basis will be a block sum of so-called elementary Jordan matrices:

$$\begin{pmatrix} \lambda_1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_1 & 1 & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & & & & \lambda_1 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \lambda_1 \end{pmatrix}$$

One can prove that this block sum is unique up to rearranging the blocks. We shall prove this in the next file of notes, but the proof is not needed for working homework exercises. However, a few definitions are needed.

Actually, these are equivalent versions of the definitions which are more convenient for working problems.

CHARACTERISTIC POLYNOMIAL
(over $\mathbb{C}$)

Let $T: V \to V$ with eigenvalues λ_j, and let $m(\lambda_j) = \dim V_{\lambda_j}$. Then

$$\chi_T(t) = \prod_j (t - \lambda_j)^{m(\lambda_j)} \cdot (-1)^n$$

$(n = \dim V)$

is the $\boxed{\text{characteristic polynomial}}$, and

$$m_T(t) = \pm \prod_j (t - \lambda_j)^{b(\lambda_j)}$$

is a/the $\boxed{\text{minimal polynomial}}$ where $b(\lambda_j)$ is the size of the largest Jordan block with λ_j's down the diagonal.

Axler, 8.50 $\quad m_T = (t-6)^2 (t-7)$

$$T: \mathbb{C}^3 \to \mathbb{C}^3.$$

The largest block for $\lambda = 6$ is 2×2, and the largest block for $\lambda = 7$ is 1×1.

Since a matrix for T is 3×3, we can only fit one 2×2 block for 6 and one 1×1 block for 7 into a Jordan form. Hence ~~the~~ a Jordan form is given by

$$\begin{pmatrix} 6 & 1 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Variant of X1 in exercises

Suppose $T: \mathbb{C}^4 \to \mathbb{C}^4$ is such that $\mathbb{C}^4 = V_\lambda$, say with $\lambda = 1$ for simplicity. What are the possibilities for the Jordan form of T?

Solution Let b_k be the number of $k\times k$ elementary Jordan blocks, so that $b_k \geq 0$ and $\sum_k k b_k = 4 = \dim \mathbb{C}^4$.

Enumerate the possibilities using a "greedy algorithm" procedure:

b_4	b_3	b_2	b_1
1	0	0	0
0	1	0	1
0	0	2	0
0	0	1	2
0	0	0	4

Start with as many big blocks as possible, then take fewer + fewer big blocks until reaching the case of all 1×1 blocks.

Hence there are 5 possibilities up to reordering the blocks.

Axler, Exercise 8C4 (made harder)

Find all Jordan forms (up to rearrangement) if $\chi = (t-1)(t-5)^3$ and $m = (t-1)(t-5)^2$.

Solution We have $T: \mathbb{C}^4 \to \mathbb{C}^4$

$\dim V_1 = 1$

$\dim V_5 = 3$

and V_5 has a 2×2 Jordan block.

V_1 is 1-dimensional $\Rightarrow$ Only one block, which is 1×1 and given by (1).

1×1 matrix

$\dim V_5 = 3$

Consider $b_1 + 2b_2 + b_3 = 3$

where $b_2 \geq 1$, $b_k \geq 0$.

The only solutions ~~are~~ is $(b_1, b_2, b_3) =$ $(1, 1, 0)$. So the Jordan blocks must be one copy each of $((5))$ and $\begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$.

1×1 matrix

More challenging $T: V \to V$ s.t.

$\dim V_0 = 3$ with $b_2 \geq 1$

$\dim V_1 = 4$ with $b_2 + b_3 + b_4 = 1$

V_0 contribution is given (as before) by $((0))$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

V_1 contribution splits into 3 cases

$b_4 = 1$ $\quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$b_3 = 1$ (so $b_1 = 1$)

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus \big((1) \big)$$

$b_2 = 1$ (so $b_1 = 2$)

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus (1) \oplus (1)$$

block sum of matrices.

$\boxed{\text{For X2, assume the matrix is } 5 \times 5.}$

Axler 8C2 $T: V \to V$ $\dim = n$ with eigenvalues 5 and 6. Prove that the characteristic polynomial divides $(5-t)^{n-1}(6-t)^{n-1}$.

Solution In the Jordan form, $\dim V_{\lambda_j} = \sum_k m(j,k)$.

So if $\dim V_{\lambda_j} = n_j$ in the preceding we have $n_j \geq 1$ but $n_1 + n_2 = n$.

Hence $n_1, n_2 \leq n-1$ and

$$\chi_T = (5-t)^{n_1}(8-t)^{n_2} \text{ divides } (5-t)^{n-1}(8-t)^{n-1}. \blacksquare$$