8F (cont'd) Applications to differential eqns.

See expmtx.pdf for background on matrix exponentials

Note Higher order systems correspond to first order systems in a larger number of variables.

Example.
$$x'' - 4y = 0 \qquad \text{Set } p = x'$$
$$y'' + 3x = 0 \qquad\qquad q = y'$$

Get a system of four eqns.

$$p' - 4y = 0 \qquad p = x'$$
$$q' + 3x = 0. \qquad q = y'$$

Similarly $\quad y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y = 0$

translates into
$$p_1 = y', \quad \text{etc.}$$
$$p_2 = p_1'$$
$$\vdots$$
$$[p_n = p_{n-1}']$$

So we get a system of n first order eqns, starting with these substitutions and concluding with $p_{n-1}' + \sum_0^{n-1} a_k p_k = 0$.

Suppose A is diagonalizable. $v_1, \dots, v_n$ basis of eigenvectors with eigenvalues $c_1, \dots, c_n$.

CLAIM: The unique solution to
$$X' = AX \quad \text{with } X(0) = v_j \text{ is}$$
$$e^{c_j t} v_j.$$

$$\frac{d}{dt} e^{c_j t} v_j = c_j v_j e^{c_j t} = A(e^{c_j t} v_j).$$

General solution ~~thenis~~ with arbitrary v.

Write $v = \sum b_j v_j$ lin comb.

Solution is $\sum b_j e^{c_j t} v_j$.

EXAMPLE Solve
$$X'(t) = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} X(t).$$
given that eigen ~~vector~~ vals are $1, 3, -2$.

First find ~~eigenvalues~~ eigenvectors.

$\lambda = 1 \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$ $\lambda = 3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $\lambda = -2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

So the general solution is

$$\begin{pmatrix} -Me^{t} + Ne^{3t} - Pe^{-2t} \\ 4Me^{t} + 2Ne^{3t} + Pe^{-2t} \\ Me^{t} + Ne^{3t} + Pe^{-2t} \end{pmatrix}$$

Note that for a complex number $a+bi$ we have $e^{(a+bi)t} = e^{at}(\cos bt + i \sin bt)$.

If ~~e^{at}~~ $a+bi$ and $a-bi$ are eigenvalues of a real matrix A, can rewrite things in terms of $e^{at}\cos bt + e^{at}\sin bt$.

<u>Example</u> $A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$

eigenvalues are $-1 \pm 2i$.

General solution $X = e^{-t}\begin{pmatrix} A\cos 2t + B\sin 2t \\ -A\sin 2t + B\cos 2t \end{pmatrix}$

Identity (not hard to prove)

If $B = P^{-1}AP$ and

$X' = BX$ has solution $Y(t)$,

then $X' = AX$ has solution $P^{-1}Y(t)$.

This follows from $\exp(B) = P^{-1}\exp(A)P$.

In some sense this reduces analysis of solutions to case of elementary J matrices

Note $\exp\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} = \begin{pmatrix} \exp B & 0 \\ 0 & \exp C \end{pmatrix}$.

Example $A = \begin{pmatrix} c & 1 \\ 0 & c \end{pmatrix} = cI + N$

$N^2 = 0$.

Since $(cI)N = N(cI)$ we have

$\exp(tcI + tN) =$

$(e^{ct}I)\cdot(I + tN)$

can forget higher terms since $N^2 = 0$.

Hence the solution with initial condition $\begin{pmatrix} u \\ v \end{pmatrix}$ is

$$e^{ct}(I + tN)\begin{pmatrix} u \\ v \end{pmatrix} = e^{ct}\begin{pmatrix} u \\ v \end{pmatrix} + te^{ct}\begin{pmatrix} v \\ 0 \end{pmatrix}.$$

For larger Jordan blocks (say $n \times n$), the matrix $\exp(I + tN)$ becomes

$$I + N + \frac{1}{2!}N^2 + \cdots + \frac{1}{(n-1)!}N^{n-1}$$

(a finite sum).

Challenge What do we get if we take

$$y^{(3)} + 3y^{(2)} + 3y' + y = 0?$$

Set up as a system of 3 first order eqns.

$$p = y'$$
$$q = y'' = p'$$
$$q' + 3q + 3p + y = 0.$$

$$\begin{pmatrix} y \\ p \\ q \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix} \begin{pmatrix} y \\ p \\ q \end{pmatrix}$$

<u>Hint</u>: The Jordan form is

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

(See if you can verify this!)

Compute $(A+I)^k$, $k = 1, 2, 3$