10A. The trace of a matrix

BACKGROUND MATERIAL

If V and W are finite dimensional vector spaces over the same scalars $\mathbb{F}$, with $n = \dim V$ and $m = \dim W$, then there is a 1-1 correspondence between linear transformations $T : V \longrightarrow W$ and $m \times n$ matrices over $\mathbb{F}$. Specifically, choose ordered bases $\mathcal{A} = \{a_1, \ldots, a_n\}$ for V and $\mathcal{B} = \{b_1, \ldots, b_m\}$ for W, then the matrix

matrix of T with respect to $\mathcal{A}, \mathcal{B}$ $\longrightarrow$ $[T]^{\mathcal{B}}_{\mathcal{A}} = (c_{ij})$ is given by

$$T a_j = \sum_{i=1}^{m} c_{ij} b_i . \qquad (1 \leq j \leq n)$$

PROPERTIES OF THE CONSTRUCTION.

If $I : V \longrightarrow V$ is the identity, then $[I]^{\mathcal{A}}_{\mathcal{A}}$ is the identity matrix.

If $S, T : V \longrightarrow W$ are linear and c is a scalar, then $[S+T]^{\mathcal{B}}_{\mathcal{A}} = [S]^{\mathcal{B}}_{\mathcal{A}} + [T]^{\mathcal{B}}_{\mathcal{A}}$ and

$$[cT]^{\mathcal{B}}_{\mathcal{A}} = c\,[T]^{\mathcal{B}}_{\mathcal{A}} .$$

If $T: V \to W$ and $U: W \to X$ are linear transformations and $\mathcal{A}, \mathcal{B}, \mathcal{D}$ are ordered bases for V, W, X respectively, then

$$[UT]_{\mathcal{A}}^{\mathcal{D}} = [U]_{\mathcal{B}}^{\mathcal{D}} \cdot [T]_{\mathcal{A}}^{\mathcal{B}}.$$

If P is an invertible $n \times n$ matrix, $\mathcal{E}$ is the ordered basis of unit vectors for $\mathbb{F}^n$, and $\mathcal{B}$ is the ordered basis given by the columns of P, then

$$P = [I]_{\mathcal{E}}^{\mathcal{B}}.$$

We have $[I]_{\mathcal{B}}^{\mathcal{A}} = \left([I]_{\mathcal{A}}^{\mathcal{B}}\right)^{-1}$ also.

Change of Basis Formula.

Let $T: V \to V$ be a linear transformation, and let $\mathcal{A}$ and $\mathcal{B}$ be ordered bases for V. If A and B are the matrices $[T]_{\mathcal{A}}^{\mathcal{A}}$ and $[T]_{\mathcal{B}}^{\mathcal{B}}$ and $P = [I]_{\mathcal{A}}^{\mathcal{B}}$, then

$$\boxed{B = PAP^{-1}.}$$

Conversely, if $B = P \cancel{\ } A P^{-1}$ where P is invertible and A is the matrix of $L_A : \mathbb{F}^n \longrightarrow \mathbb{F}^n$, $L_A(x) = Ax$, then

$$B = [L_A]_{\beta}^{\beta} = PAP^{-1}$$

where β is the ordered basis whose ~~eley~~ vectors are the columns of P.

Derivation. (First part)

$$B = [T]_{\mathcal{B}}^{\mathcal{B}} = [IT]_{\mathcal{B}}^{\mathcal{B}} = [I]_{\mathcal{A}}^{\mathcal{B}} [T]_{\mathcal{B}}^{\mathcal{A}} =$$

$$[I]_{\mathcal{A}}^{\mathcal{B}} [T]_{\mathcal{A}}^{\mathcal{A}} [I]_{\mathcal{B}}^{\mathcal{A}} = PAP^{-1}. \blacksquare$$

Second part Note that $A = [L_A]_{\mathcal{E}}^{\mathcal{E}}$, so $B =$

$$[L_A]_{\beta}^{\beta} = [I]_{\mathcal{E}}^{\beta} [L_A]_{\mathcal{E}}^{\mathcal{E}} [I]_{\beta}^{\mathcal{E}} = PAP^{-1}. \blacksquare$$

Definition Two $n \times n$ matrices A and B are similar if there is an invertible matrix P such that $B = PAP^{-1}$.

Properties of matrix similarity

A is similar to A, for $A = I \cdot A \cdot \underbrace{I^{-1}}_{= I}$

If A is similar to B, then B is similar to A, for $B = PAP^{-1} \Rightarrow A = P^{-1}BP = P^{-1}B(P^{-1})^{-1}$.

If A is similar to B and B is similar to C, then A is similar to C, for $B = PAP^{-1}$, $C = QBQ^{-1}$

$\Rightarrow C = Q(PAP^{-1})Q^{-1} = (QP)A(QP)^{-1}$.

> Two matrices are similar $\Leftrightarrow$ they represent the same linear transformation with respect to different bases.

Definition of the trace

Let A be an $n \times n$ matrix. Then the $\boxed{\text{TRACE}}$ of A,

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Notice that tr is a linear transformation from matrices to the scalars.

PROPOSITION. If A and B are $n\times n$ matrices, then $tr(AB) = tr(BA)$.

COROLLARY If P is invertible, then $tr(PAP^{-1}) = tr(A)$ [similar matrices have the same trace].

Derivation of the Corollary.

$$tr(PAP^{-1}) = tr(P^{-1}(PA)) = tr(P^{-1}PA) = tr(A). \blacksquare$$

Proof of the Proposition.

$$tr(AB) = \sum_i (AB)_{ii} = \sum_i \sum_j a_{ij} b_{ji} = \sum_j \sum_i a_{ij} b_{ji} = \sum_j \sum_i b_{ji} a_{ij} = \sum_j (BA)_{jj} = tr(BA). \blacksquare$$

THEOREM. Let A be $n\times n$, and let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of $L_A : \mathbb{C}^n \to \mathbb{C}^n$. If m_i is the dimension of V_{λ_i}, then $tr(A) = \sum m_i \lambda_i$.

<u>PROOF.</u> We know $A = PBP^{-1}$ where B is in Jordan form, so $tr(A) = tr(B)$. But if B is in Jordan form, then $tr(B)$ is given by the formula in the conclusion. Finally, B and A have the same eigenvalues, ~~and~~ for $Ax = \lambda x \implies$ ~~B(P~~

$$B(P^{-1}x) = P^{-1}AP(P^{-1}x) = P^{-1}Ax =$$

$P^{-1}(\lambda x) = \lambda P^{-1}x$, and similarly

$$V_\lambda(B) = P^{-1}[V_\lambda(A)]. \blacksquare$$

<u>"Application."</u> Suppose A is a 3×3 matrix such that (i) 3 and 5 are eigenvalues, (ii) $tr(A) = 12$. What is the third eigenvalue?

<u>Solution</u> $12 = tr(A) = \lambda_1 + \lambda_2 + \lambda_3 = 3 + 5 + \lambda_3$, so $\lambda_3 = 4$.