USING CHARACTERISTIC POLY. TO FIND E'VALS.

<u>Example</u> Find the eigenvalues for

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{pmatrix}$$

Since λ is an eigenvalue
$\iff A-\lambda I$ not invertible
$\iff \det(A-\lambda I)=0$,

look at the characteristic polynomial:

$$\det(A-xI) = \det \begin{pmatrix} 2-x & 1 & 0 \\ -1 & -x & 1 \\ 1 & 3 & 1-x \end{pmatrix}$$

Expand this by minors along 1st row:

$$(2-x)\begin{vmatrix} -x & 1 \\ 3 & 1-x \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ 1 & 1-x \end{vmatrix} =$$

$$-(x-2)(x^2-x-2) = -(x-2)^2(x+1).$$

So the eigenvalues are 2 and -1.

Is there a basis

$$A-2I = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & 1 \\ 1 & 3 & 1 \end{pmatrix}$$

This has rank 2, so eigenspace is 1-dim.

It follows that the Jordan form is $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

<u>Another example</u>

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}$$

The characteristic polynomial is

$$\begin{vmatrix} 1-x & 2 & -1 \\ 1 & -x & 1 \\ 4 & 4 & 5-x \end{vmatrix} = -x^3 + 6x^2 - 11x + 6$$

so the eigenvalues are the roots of

$$x^3 - 6x^2 + 11x - 6 = 0$$

One problem is whether there are any roots that are easy to describe algebraically; for example, as rational numbers.

<u>Theorem of Gauss.</u> If a monic polynomial (leading term coeff $= 1$) has a rational root, ~~this root must be an integer.~~ and its coefficients are integers, the root must be an integer.

Furthermore, the root (evenly) divides the constant term.

Check for integral roots. The only possibilities are $\pm 1, \pm 2, \pm 3, \pm 6$.
Evaluate the polynomial at each of these.
The roots turn out to be $1, 2, 3$.

<u>"Bad" example</u>

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

Characteristic polynomial is

$$(1-x)(2-x)(3-x) + 4 - (2-x) - 4(1-x) - (3-x)$$
$$= 1 - 5x + 6x^2 - x^3.$$

Only possible rational roots are ± 1.
Neither is a root of the characteristic poly.
Hence 3 irrational roots. None are easy to describe algebraically, so we must settle for numerical approximations.

"Half bad" example

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

Char. poly is $-x^3 + 5x^2 - 7x + 2$.

Possible roots: $\pm 1, \pm 2$. Check 2 is a root.

Factor characteristic polynomial as

$$(2-x)(1-3x+x^2).$$

Roots of the quadratic polynomial are

$$\frac{3 \pm \sqrt{5}}{2}. \blacksquare$$