(iii) If I is the 2×2 identity matrix, then $\delta(I) = 1$.

Prove that $\delta(A) = \det(A)$ for all $A \in \mathsf{M}_{2\times 2}(F)$. (This result is generalized in Section 4.5.)

12. Let $\{u, v\}$ be an ordered basis for R^2. Prove that

$$\mathsf{O}\begin{pmatrix} u \\ v \end{pmatrix} = 1$$

if and only if $\{u, v\}$ forms a right-handed coordinate system. *Hint:* Recall the definition of a rotation given in Example 2 of Section 2.1.

4.2 DETERMINANTS OF ORDER n

In this section, we extend the definition of the determinant to $n \times n$ matrices for $n \geq 3$. For this definition, it is convenient to introduce the following notation: Given $A \in \mathsf{M}_{n\times n}(F)$, for $n \geq 2$, denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j by $\tilde{A}_{ij}$. Thus for

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \in \mathsf{M}_{3\times 3}(R),$$

we have

$$\tilde{A}_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}, \qquad \tilde{A}_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}, \quad \text{and} \quad \tilde{A}_{32} = \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix},$$

and for

$$B = \begin{pmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{pmatrix} \in \mathsf{M}_{4\times 4}(R),$$

we have

$$\tilde{B}_{23} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -5 & 8 \\ -2 & 6 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{B}_{42} = \begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & -1 \\ 2 & -3 & 8 \end{pmatrix}.$$

Definitions. *Let $A \in \mathsf{M}_{n\times n}(F)$. If $n = 1$, so that $A = (A_{11})$, we define* $\det(A) = A_{11}$. *For $n \geq 2$, we define $\det(A)$ recursively as*

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \det(\tilde{A}_{1j}).$$

The scalar $\det(A)$ *is called the* **determinant** *of* A *and is also denoted by* $|A|$. *The scalar*

$$(-1)^{i+j}\det(\tilde{A}_{ij})$$

is called the **cofactor** *of the entry of* A *in row* i, *column* j.

Letting

$$c_{ij} = (-1)^{i+j}\det(\tilde{A}_{ij})$$

denote the cofactor of the row i, column j entry of A, we can express the formula for the determinant of A as

$$\det(A) = A_{11}c_{11} + A_{12}c_{12} + \cdots + A_{1n}c_{1n}.$$

Thus the determinant of A equals the sum of the products of each entry in row 1 of A multiplied by its cofactor. This formula is called **cofactor expansion along the first row** of A. Note that, for 2×2 matrices, this definition of the determinant of A agrees with the one given in Section 4.1 because

$$\det(A) = A_{11}(-1)^{1+1}\det(\tilde{A}_{11}) + A_{12}(-1)^{1+2}\det(\tilde{A}_{12}) = A_{11}A_{22} - A_{12}A_{21}.$$

Example 1

Let

$$A = \begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix} \in \mathsf{M}_{3\times 3}(R).$$

Using cofactor expansion along the first row of A, we obtain

$$\begin{aligned}
\det(A) &= (-1)^{1+1}A_{11}\cdot\det(\tilde{A}_{11}) + (-1)^{1+2}A_{12}\cdot\det(\tilde{A}_{12}) \\
&\quad + (-1)^{1+3}A_{13}\cdot\det(\tilde{A}_{13}) \\
&= (-1)^2(1)\cdot\det\begin{pmatrix} -5 & 2 \\ 4 & -6 \end{pmatrix} + (-1)^3(3)\cdot\det\begin{pmatrix} -3 & 2 \\ -4 & -6 \end{pmatrix} \\
&\quad + (-1)^4(-3)\cdot\det\begin{pmatrix} -3 & -5 \\ -4 & 4 \end{pmatrix} \\
&= 1\,[-5(-6) - 2(4)] - 3\,[-3(-6) - 2(-4)] - 3\,[-3(4) - (-5)(-4)] \\
&= 1(22) - 3(26) - 3(-32) \\
&= 40. \quad \blacklozenge
\end{aligned}$$

Example 2

Let

$$B = \begin{pmatrix} 0 & 1 & 3 \\ -2 & -3 & -5 \\ 4 & -4 & 4 \end{pmatrix} \in \mathsf{M}_{3\times 3}(R).$$

Using cofactor expansion along the first row of B, we obtain

$$\begin{aligned}
\det(B) &= (-1)^{1+1}B_{11}\cdot\det(\tilde{B}_{11}) + (-1)^{1+2}B_{12}\cdot\det(\tilde{B}_{12}) \\
&\quad + (-1)^{1+3}B_{13}\cdot\det(\tilde{B}_{13}) \\
&= (-1)^2(0)\cdot\det\begin{pmatrix} -3 & -5 \\ -4 & 4 \end{pmatrix} + (-1)^3(1)\cdot\det\begin{pmatrix} -2 & -5 \\ 4 & 4 \end{pmatrix} \\
&\quad + (-1)^4(3)\cdot\det\begin{pmatrix} -2 & -3 \\ 4 & -4 \end{pmatrix} \\
&= 0 - 1\,[-2(4) - (-5)(4)] + 3\,[-2(-4) - (-3)(4)] \\
&= 0 - 1(12) + 3(20) \\
&= 48. \quad \blacklozenge
\end{aligned}$$

Example 3

Let

$$C = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{pmatrix} \in \mathsf{M}_{4\times 4}(R).$$

Using cofactor expansion along the first row of C and the results of Examples 1 and 2, we obtain

$$\begin{aligned}
\det(C) &= (-1)^2(2)\cdot\det(\tilde{C}_{11}) + (-1)^3(0)\cdot\det(\tilde{C}_{12}) \\
&\quad + (-1)^4(0)\cdot\det(\tilde{C}_{13}) + (-1)^5(1)\cdot\det(\tilde{C}_{14}) \\
&= (-1)^2(2)\cdot\det\begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix} + 0 + 0 \\
&\quad + (-1)^5(1)\cdot\det\begin{pmatrix} 0 & 1 & 3 \\ -2 & -3 & -5 \\ 4 & -4 & 4 \end{pmatrix} \\
&= 2(40) + 0 + 0 - 1(48) \\
&= 32. \quad \blacklozenge
\end{aligned}$$

Example 4

The determinant of the $n \times n$ identity matrix is 1. We prove this assertion by mathematical induction on n. The result is clearly true for the 1×1 identity matrix. Assume that the determinant of the $(n-1) \times (n-1)$ identity matrix is 1 for some $n \geq 2$, and let I denote the $n \times n$ identity matrix. Using cofactor expansion along the first row of I, we obtain

$$\begin{aligned}\det(I) &= (-1)^2(1)\cdot\det(\tilde{I}_{11}) + (-1)^3(0)\cdot\det(\tilde{I}_{12}) + \cdots \\ &\qquad + (-1)^{1+n}(0)\cdot\det(\tilde{I}_{1n}) \\ &= 1(1) + 0 + \cdots + 0 \\ &= 1\end{aligned}$$

because $\tilde{I}_{11}$ is the $(n-1) \times (n-1)$ identity matrix. This shows that the determinant of the $n \times n$ identity matrix is 1, and so the determinant of any identity matrix is 1 by the principle of mathematical induction. ♦

As is illustrated in Example 3, the calculation of a determinant using the recursive definition is extremely tedious, even for matrices as small as 4×4. Later in this section, we present a more efficient method for evaluating determinants, but we must first learn more about them.

Recall from Theorem 4.1 (p. 200) that, although the determinant of a 2×2 matrix is *not* a linear transformation, it is a linear function of each row when the other row is held fixed. We now show that a similar property is true for determinants of any size.

Theorem 4.3. *The determinant of an $n \times n$ matrix is a linear function of each row when the remaining rows are held fixed. That is, for $1 \leq r \leq n$, we have*

$$\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u + kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

whenever k is a scalar and u, v, and each a_i are row vectors in F^n.

Proof. The proof is by mathematical induction on n. The result is immediate if $n = 1$. Assume that for some integer $n \geq 2$ the determinant of any $(n-1) \times (n-1)$ matrix is a linear function of each row when the remaining rows are held fixed. Let A be an $n \times n$ matrix with rows $a_1, a_2, \ldots, a_n$, respectively, and suppose that for some r $(1 \le r \le n)$, we have $a_r = u + kv$ for some $u, v \in \mathsf{F}^n$ and some scalar k. Let $u = (b_1, b_2, \ldots, b_n)$ and $v = (c_1, c_2, \ldots, c_n)$, and let B and C be the matrices obtained from A by replacing row r of A by u and v, respectively. We must prove that $\det(A) = \det(B) + k\det(C)$. We leave the proof of this fact to the reader for the case $r = 1$. For $r > 1$ and $1 \le j \le n$, the rows of $\tilde{A}_{1j}$, $\tilde{B}_{1j}$, and $\tilde{C}_{1j}$ are the same except for row $r - 1$. Moreover, row $r - 1$ of $\tilde{A}_{1j}$ is

$$(b_1 + kc_1, \ldots, b_{j-1} + kc_{j-1}, b_{j+1} + kc_{j+1}, \ldots, b_n + kc_n),$$

which is the sum of row $r-1$ of $\tilde{B}_{1j}$ and k times row $r-1$ of $\tilde{C}_{1j}$. Since $\tilde{B}_{1j}$ and $\tilde{C}_{1j}$ are $(n-1) \times (n-1)$ matrices, we have

$$\det(\tilde{A}_{1j}) = \det(\tilde{B}_{1j}) + k\det(\tilde{C}_{1j})$$

by the induction hypothesis. Thus since $A_{1j} = B_{1j} = C_{1j}$, we have

$$\begin{aligned}
\det(A) &= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \det(\tilde{A}_{1j}) \\
&= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \left[\det(\tilde{B}_{1j}) + k\det(\tilde{C}_{1j})\right] \\
&= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \det(\tilde{B}_{1j}) + k \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \det(\tilde{C}_{1j}) \\
&= \det(B) + k\det(C).
\end{aligned}$$

This shows that the theorem is true for $n \times n$ matrices, and so the theorem is true for all square matrices by mathematical induction. ∎

Corollary. *If $A \in \mathsf{M}_{n\times n}(F)$ has a row consisting entirely of zeros, then* $\det(A) = 0$.

Proof. See Exercise 24. ∎

The definition of a determinant requires that the determinant of a matrix be evaluated by cofactor expansion along the first row. Our next theorem shows that the determinant of a square matrix can be evaluated by cofactor expansion along *any* row. Its proof requires the following technical result.

Lemma. *Let $B \in \mathsf{M}_{n\times n}(F)$, where $n \ge 2$. If row i of B equals e_k for some k $(1 \le k \le n)$, then* $\det(B) = (-1)^{i+k}\det(\tilde{B}_{ik})$.

Proof. The proof is by mathematical induction on n. The lemma is easily proved for $n = 2$. Assume that for some integer $n \geq 3$, the lemma is true for $(n-1) \times (n-1)$ matrices, and let B be an $n \times n$ matrix in which row i of B equals e_k for some k $(1 \leq k \leq n)$. The result follows immediately from the definition of the determinant if $i = 1$. Suppose therefore that $1 < i \leq n$. For each $j \neq k$ $(1 \leq j \leq n)$, let C_{ij} denote the $(n-2) \times (n-2)$ matrix obtained from B by deleting rows 1 and i and columns j and k. For each j, row $i-1$ of $\tilde{B}_{1j}$ is the following vector in F^{n-1}:

$$\begin{cases} e_{k-1} & \text{if } j < k \\ \mathit{0} & \text{if } j = k \\ e_k & \text{if } j > k. \end{cases}$$

Hence by the induction hypothesis and the corollary to Theorem 4.3, we have

$$\det(\tilde{B}_{1j}) = \begin{cases} (-1)^{(i-1)+(k-1)} \det(C_{ij}) & \text{if } j < k \\ 0 & \text{if } j = k \\ (-1)^{(i-1)+k} \det(C_{ij}) & \text{if } j > k. \end{cases}$$

Therefore

$$\begin{aligned}
\det(B) &= \sum_{j=1}^{n} (-1)^{1+j} B_{1j} \cdot \det(\tilde{B}_{1j}) \\
&= \sum_{j<k} (-1)^{1+j} B_{1j} \cdot \det(\tilde{B}_{1j}) + \sum_{j>k} (-1)^{1+j} B_{1j} \cdot \det(\tilde{B}_{1j}) \\
&= \sum_{j<k} (-1)^{1+j} B_{1j} \cdot \left[(-1)^{(i-1)+(k-1)} \det(C_{ij})\right] \\
&\qquad + \sum_{j>k} (-1)^{1+j} B_{1j} \cdot \left[(-1)^{(i-1)+k} \det(C_{ij})\right] \\
&= (-1)^{i+k} \left[\sum_{j<k} (-1)^{1+j} B_{1j} \cdot \det(C_{ij}) \right. \\
&\qquad \left. + \sum_{j>k} (-1)^{1+(j-1)} B_{1j} \cdot \det(C_{ij}) \right].
\end{aligned}$$

Because the expression inside the preceding bracket is the cofactor expansion of $\tilde{B}_{ik}$ along the first row, it follows that

$$\det(B) = (-1)^{i+k} \det(\tilde{B}_{ik}).$$

This shows that the lemma is true for $n \times n$ matrices, and so the lemma is true for all square matrices by mathematical induction. ∎

We are now able to prove that cofactor expansion along any row can be used to evaluate the determinant of a square matrix.

Theorem 4.4. *The determinant of a square matrix can be evaluated by cofactor expansion along any row. That is, if $A \in \mathsf{M}_{n\times n}(F)$, then for any integer i $(1 \leq i \leq n)$,*

$$\det(A) = \sum_{j=1}^{n}(-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}).$$

Proof. Cofactor expansion along the first row of A gives the determinant of A by definition. So the result is true if $i = 1$. Fix $i > 1$. Row i of A can be written as $\sum_{j=1}^{n} A_{ij}e_j$. For $1 \leq j \leq n$, let B_j denote the matrix obtained from A by replacing row i of A by e_j. Then by Theorem 4.3 and the lemma, we have

$$\det(A) = \sum_{j=1}^{n} A_{ij} \det(B_j) = \sum_{j=1}^{n}(-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}). \qquad \blacksquare$$

Corollary. *If $A \in \mathsf{M}_{n\times n}(F)$ has two identical rows, then* $\det(A) = 0$.

Proof. The proof is by mathematical induction on n. We leave the proof of the result to the reader in the case that $n = 2$. Assume that for some integer $n \geq 3$, it is true for $(n-1) \times (n-1)$ matrices, and let rows r and s of $A \in \mathsf{M}_{n\times n}(F)$ be identical for $r \neq s$. Because $n \geq 3$, we can choose an integer i $(1 \leq i \leq n)$ other than r and s. Now

$$\det(A) = \sum_{j=1}^{n}(-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij})$$

by Theorem 4.4. Since each $\tilde{A}_{ij}$ is an $(n-1) \times (n-1)$ matrix with two identical rows, the induction hypothesis implies that each $\det(\tilde{A}_{ij}) = 0$, and hence $\det(A) = 0$. This completes the proof for $n \times n$ matrices, and so the lemma is true for all square matrices by mathematical induction. $\blacksquare$

It is possible to evaluate determinants more efficiently by combining cofactor expansion with the use of elementary row operations. Before such a process can be developed, we need to learn what happens to the determinant of a matrix if we perform an elementary row operation on that matrix. Theorem 4.3 provides this information for elementary row operations of type 2 (those in which a row is multiplied by a nonzero scalar). Next we turn our attention to elementary row operations of type 1 (those in which two rows are interchanged).

Theorem 4.5. *If $A \in \mathsf{M}_{n\times n}(F)$ and B is a matrix obtained from A by interchanging any two rows of A, then* $\det(B) = -\det(A)$.

Proof. Let the rows of $A \in \mathsf{M}_{n\times n}(F)$ be $a_1, a_2, \ldots, a_n$, and let B be the matrix obtained from A by interchanging rows r and s, where $r < s$. Thus

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_s \\ \vdots \\ a_n \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix}.$$

Consider the matrix obtained from A by replacing rows r and s by $a_r + a_s$. By the corollary to Theorem 4.4 and Theorem 4.3, we have

$$0 = \det\begin{pmatrix} a_1 \\ \vdots \\ a_r + a_s \\ \vdots \\ a_r + a_s \\ \vdots \\ a_n \end{pmatrix} = \det\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_r + a_s \\ \vdots \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r + a_s \\ \vdots \\ a_n \end{pmatrix}$$

$$= \det\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_s \\ \vdots \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_s \\ \vdots \\ a_n \end{pmatrix}$$

$$= 0 + \det(A) + \det(B) + 0.$$

Therefore $\det(B) = -\det(A)$. ∎

We now complete our investigation of how an elementary row operation affects the determinant of a matrix by showing that elementary row operations of type 3 do not change the determinant of a matrix.

Theorem 4.6. *Let $A \in \mathsf{M}_{n\times n}(F)$, and let B be a matrix obtained by adding a multiple of one row of A to another row of A. Then* $\det(B) = \det(A)$.

Proof. Suppose that B is the $n \times n$ matrix obtained from A by adding k times row r to row s, where $r \neq s$. Let the rows of A be $a_1, a_2, \ldots, a_n$, and the rows of B be $b_1, b_2, \ldots, b_n$. Then $b_i = a_i$ for $i \neq s$ and $b_s = a_s + ka_r$. Let C be the matrix obtained from A by replacing row s with a_r. Applying Theorem 4.3 to row s of B, we obtain

$$\det(B) = \det(A) + k\det(C) = \det(A)$$

because $\det(C) = 0$ by the corollary to Theorem 4.4. ∎

In Theorem 4.2 (p. 201), we proved that a 2×2 matrix is invertible if and only if its determinant is nonzero. As a consequence of Theorem 4.6, we can prove half of the promised generalization of this result in the following corollary. The converse is proved in the corollary to Theorem 4.7.

Corollary. *If* $A \in \mathsf{M}_{n\times n}(F)$ *has rank less than* n, *then* $\det(A) = 0$.

Proof. If the rank of A is less than n, then the rows $a_1, a_2, \ldots, a_n$ of A are linearly dependent. By Exercise 14 of Section 1.5, some row of A, say, row r, is a linear combination of the other rows. So there exist scalars c_i such that

$$a_r = c_1a_1 + \cdots + c_{r-1}a_{r-1} + c_{r+1}a_{r+1} + \cdots + c_na_n.$$

Let B be the matrix obtained from A by adding $-c_i$ times row i to row r for each $i \neq r$. Then row r of B consists entirely of zeros, and so $\det(B) = 0$. But by Theorem 4.6, $\det(B) = \det(A)$. Hence $\det(A) = 0$. ∎

The following rules summarize the effect of an elementary row operation on the determinant of a matrix $A \in \mathsf{M}_{n\times n}(F)$.

(a) If B is a matrix obtained by interchanging any two rows of A, then $\det(B) = -\det(A)$.
(b) If B is a matrix obtained by multiplying a row of A by a nonzero scalar k, then $\det(B) = k\det(A)$.
(c) If B is a matrix obtained by adding a multiple of one row of A to another row of A, then $\det(B) = \det(A)$.

These facts can be used to simplify the evaluation of a determinant. Consider, for instance, the matrix in Example 1:

$$A = \begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix}.$$

Adding 3 times row 1 of A to row 2 and 4 times row 1 to row 3, we obtain

$$M = \begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & -7 \\ 0 & 16 & -18 \end{pmatrix}.$$

Since M was obtained by performing two type 3 elementary row operations on A, we have $\det(A) = \det(M)$. The cofactor expansion of M along the first row gives

$$\det(M) = (-1)^{1+1}(1)\cdot\det(\tilde{M}_{11}) + (-1)^{1+2}(3)\cdot\det(\tilde{M}_{12}) + (-1)^{1+3}(-3)\cdot\det(\tilde{M}_{13}).$$

Both $\tilde{M}_{12}$ and $\tilde{M}_{13}$ have a column consisting entirely of zeros, and so $\det(\tilde{M}_{12}) = \det(\tilde{M}_{13}) = 0$ by the corollary to Theorem 4.6. Hence

$$\begin{aligned}\det(M) &= (-1)^{1+1}(1)\cdot\det(\tilde{M}_{11})\\ &= (-1)^{1+1}(1)\cdot\det\begin{pmatrix}4 & -7\\ 16 & -18\end{pmatrix}\\ &= 1[4(-18) - (-7)(16)] = 40.\end{aligned}$$

Thus with the use of two elementary row operations of type 3, we have reduced the computation of $\det(A)$ to the evaluation of one determinant of a 2×2 matrix.

But we can do even better. If we add -4 times row 2 of M to row 3 (another elementary row operation of type 3), we obtain

$$P = \begin{pmatrix}1 & 3 & -3\\ 0 & 4 & -7\\ 0 & 0 & 10\end{pmatrix}.$$

Evaluating $\det(P)$ by cofactor expansion along the first row, we have

$$\begin{aligned}\det(P) &= (-1)^{1+1}(1)\cdot\det(\tilde{P}_{11})\\ &= (-1)^{1+1}(1)\cdot\det\begin{pmatrix}4 & -7\\ 0 & 10\end{pmatrix} = 1\cdot 4\cdot 10 = 40,\end{aligned}$$

as described earlier. Since $\det(A) = \det(M) = \det(P)$, it follows that $\det(A) = 40$.

The preceding calculation of $\det(P)$ illustrates an important general fact. *The determinant of an upper triangular matrix is the product of its diagonal entries.* (See Exercise 23.) By using elementary row operations of types 1 and 3 only, we can transform any square matrix into an upper triangular matrix, and so we can easily evaluate the determinant of any square matrix. The next two examples illustrate this technique.

Example 5

To evaluate the determinant of the matrix

$$B = \begin{pmatrix}0 & 1 & 3\\ -2 & -3 & -5\\ 4 & -4 & 4\end{pmatrix}$$

in Example 2, we must begin with a row interchange. Interchanging rows 1 and 2 of B produces

$$C = \begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 4 & -4 & 4 \end{pmatrix}.$$

By means of a sequence of elementary row operations of type 3, we can transform C into an upper triangular matrix:

$$\begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 4 & -4 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 0 & -10 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 24 \end{pmatrix}.$$

Thus $\det(C) = -2 \cdot 1 \cdot 24 = -48$. Since C was obtained from B by an interchange of rows, it follows that

$$\det(B) = -\det(C) = 48. \quad \blacklozenge$$

Example 6

The technique in Example 5 can be used to evaluate the determinant of the matrix

$$C = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{pmatrix}$$

in Example 3. This matrix can be transformed into an upper triangular matrix by means of the following sequence of elementary row operations of type 3:

$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & -3 & -5 & 3 \\ 0 & -4 & 4 & -8 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -6 \\ 0 & 0 & 16 & -20 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Thus $\det(C) = 2 \cdot 1 \cdot 4 \cdot 4 = 32$. $\blacklozenge$

Using elementary row operations to evaluate the determinant of a matrix, as illustrated in Example 6, is far more efficient than using cofactor expansion. Consider first the evaluation of a 2×2 matrix. Since

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc,$$

the evaluation of the determinant of a 2×2 matrix requires 2 multiplications (and 1 subtraction). For $n \geq 3$, evaluating the determinant of an $n \times n$ matrix by cofactor expansion along any row expresses the determinant as a sum of n products involving determinants of $(n-1) \times (n-1)$ matrices. Thus in all, the evaluation of the determinant of an $n \times n$ matrix by cofactor expansion along any row requires over $n!$ multiplications, whereas evaluating the determinant of an $n \times n$ matrix by elementary row operations as in Examples 5 and 6 can be shown to require only $(n^3 + 2n - 3)/3$ multiplications. To evaluate the determinant of a 20×20 matrix, which is not large by present standards, cofactor expansion along a row requires over $20! \approx 2.4 \times 10^{18}$ multiplications. Thus it would take a computer performing one billion multiplications per second over 77 years to evaluate the determinant of a 20×20 matrix by this method. By contrast, the method using elementary row operations requires only 2679 multiplications for this calculation and would take the same computer less than three-millionths of a second! It is easy to see why most computer programs for evaluating the determinant of an arbitrary matrix do not use cofactor expansion.

In this section, we have defined the determinant of a square matrix in terms of cofactor expansion along the first row. We then showed that the determinant of a square matrix can be evaluated using cofactor expansion along *any* row. In addition, we showed that the determinant possesses a number of special properties, including properties that enable us to calculate $\det(B)$ from $\det(A)$ whenever B is a matrix obtained from A by means of an elementary row operation. These properties enable us to evaluate determinants much more efficiently. In the next section, we continue this approach to discover additional properties of determinants.

EXERCISES

1. Label the following statements as true or false.

(a) The function $\det\colon \mathsf{M}_{n\times n}(F) \to F$ is a linear transformation.
(b) The determinant of a square matrix can be evaluated by cofactor expansion along any row.
(c) If two rows of a square matrix A are identical, then $\det(A) = 0$.
(d) If B is a matrix obtained from a square matrix A by interchanging any two rows, then $\det(B) = -\det(A)$.
(e) If B is a matrix obtained from a square matrix A by multiplying a row of A by a scalar, then $\det(B) = \det(A)$.
(f) If B is a matrix obtained from a square matrix A by adding k times row i to row j, then $\det(B) = k\det(A)$.
(g) If $A \in \mathsf{M}_{n\times n}(F)$ has rank n, then $\det(A) = 0$.
(h) The determinant of an upper triangular matrix equals the product of its diagonal entries.