37. (*Application to permutation theory*) Consider an arrangement of n objects, lined up in a column. A rearrangement of the order of the objects is called a *permutation* of the objects. Every such permutation can be achieved by successively swapping the positions of pairs of the objects. For example, the first swap might be to interchange the first object with whatever one you want to be first in the new arrangement, and then continuing this procedure with the second, the third, etc. However, there are many possible sequences of swaps that will achieve a given permutation. Use the theory of determinants to prove that it is impossible to achieve the same permutation using both an even number and an odd number of swaps. [HINT: It doesn't matter what the objects actually are—think of them as being the rows of an $n \times n$ matrix.]

38. This exercise is for the reader who is skeptical of our assertion that the solar system would be dead long before a present-day computer could find the determinant of a 50×50 matrix using just Definition 4.1 with expansion by minors.
    a. Recall that $n! = n(n-1) \cdots (3)(2)(1)$. Show by induction that expansion of an $n \times n$ matrix by minors requires at least $n!$ multiplications for $n > 1$.
    b. Run the routine EBYMTIME in LINTEK and find the time required to perform $n!$ multiplications for $n = 8, 12, 16, 20, 25, 30, 40, 50, 70$, and 100.

39. Use MATLAB or the routine MATCOMP in LINTEK to check Example 2 and Exercises 5–10. Load the appropriate file of matrices if it is accessible. The determinant of a matrix A is found in MATLAB using the command **det(A)**.

4.3 COMPUTATION OF DETERMINANTS AND CRAMER'S RULE

We have seen that computation of determinants of high order is an unreasonable task if it is done directly from Definition 4.1, relying entirely on repeated expansion by minors. In the special case where a square matrix is triangular, Example 4 in Section 4.2 shows that the determinant is simply the product of the diagonal entries. We know that a matrix can be reduced to row-echelon form by means of elementary row operations, and row-echelon form for a square matrix is always triangular. The discussion leading to Theorem 4.3 in the previous section actually shows how the determinant of a matrix can be computed by a row reduction to echelon form. We rephrase part of this discussion in a box as an algorithm that a computer might follow to find a determinant.

Computation of a Determinant

The determinant of an $n \times n$ matrix A can be computed as follows:

1. Reduce A to an echelon form, using only row addition and row interchanges.
2. If any of the matrices appearing in the reduction contains a row of zeros, then $\det(A) = 0$.

3. Otherwise,

$$\det(A) = (-1)^r \cdot (\text{Product of pivots}),$$

where r is the number of row interchanges performed.

When doing a computation with pencil and paper rather than with a computer, we often use row scaling to make pivots 1, in order to ease calculations. As you study the following example, notice how the pivots accumulate as factors when the scalar-multiplication property of determinants is repeatedly used.

EXAMPLE 1 Find the determinant of the following matrix by reducing it to row-echelon form.

$$A = \begin{bmatrix} 2 & 2 & 0 & 4 \\ 3 & 3 & 2 & 2 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 2 & 1 \end{bmatrix}$$

SOLUTION We find that

$$\begin{vmatrix} 2 & 2 & 0 & 4 \\ 3 & 3 & 2 & 2 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 0 & 2 \\ 3 & 3 & 2 & 2 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 2 & 1 \end{vmatrix} \qquad \textbf{Scalar-multiplication property}$$

$$= 2\begin{vmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & -4 \\ 0 & 1 & 3 & 2 \\ 0 & -2 & 2 & -3 \end{vmatrix} \qquad \textbf{Row-addition property twice}$$

$$= -2\begin{vmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 2 & -4 \\ 0 & -2 & 2 & -3 \end{vmatrix} \qquad \textbf{Row-interchange property}$$

$$= -2\begin{vmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 8 & 1 \end{vmatrix} \qquad \textbf{Row-addition property}$$

$$= (-2)(2)\begin{vmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 8 & 1 \end{vmatrix} \qquad \textbf{Scalar-multiplication property}$$

$$= (-2)(2)\begin{vmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 17 \end{vmatrix}. \quad \text{Row-addition property}$$

Therefore, $\det(A) = (-2)(2)(17) = -68$. ▪

In our written work, we usually don't write out the shaded portion of the computation in the preceding example.

Row reduction offers an efficient way to program a computer to compute a determinant. If we are using pencil and paper, a further modification is more practical. We can use elementary row or column operations and the properties of determinants to reduce the computation to the determinant of a matrix having some row or column with a sole nonzero entry. A computer program generally modifies the matrix so that the first column has a single nonzero entry, but we can look at the matrix and choose the row or column where this can be achieved most easily. Expanding by minors on that row or column reduces the computation to a determinant of order one less, and we can continue the process until we are left with the computation of a determinant of a 2×2 matrix. Here is an illustration.

EXAMPLE 2 Find the determinant of the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 2 & 0 & 1 & 0 \\ 6 & 1 & 3 & 4 \\ -7 & 3 & -2 & 8 \end{bmatrix}.$$

SOLUTION It is easiest to create zeros in the second row and then expand by minors on that row. We start by adding -2 times the third column to the first column, and we continue in this fashion:

$$\begin{vmatrix} 2 & -1 & 3 & 5 \\ 2 & 0 & 1 & 0 \\ 6 & 1 & 3 & 4 \\ -7 & 3 & -2 & 8 \end{vmatrix} = \begin{vmatrix} -4 & -1 & 3 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 4 \\ -3 & 3 & -2 & 8 \end{vmatrix} = -\begin{vmatrix} -4 & -1 & 5 \\ 0 & 1 & 4 \\ -3 & 3 & 8 \end{vmatrix}$$

$$= -\begin{vmatrix} -4 & -1 & 9 \\ 0 & 1 & 0 \\ -3 & 3 & -4 \end{vmatrix} = -\begin{vmatrix} -4 & 9 \\ -3 & -4 \end{vmatrix}$$

$$= -(16 + 27) = -43.$$ ▪

Cramer's Rule

We now exhibit formulas in terms of determinants for the components in the solution vector of a square linear system $A\mathbf{x} = \mathbf{b}$, where A is an invertible matrix. The formulas are contained in the following theorem.

THEOREM 4.5 Cramer's Rule

Consider the linear system $A\mathbf{x} = \mathbf{b}$, where $A = [a_{ij}]$ is an $n \times n$ invertible matrix,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

The system has a unique solution given by

$$x_k = \frac{\det(B_k)}{\det(A)} \quad \text{for} \quad k = 1, \ldots, n, \tag{1}$$

where B_k is the matrix obtained from A by replacing the kth-column vector of A by the column vector $\mathbf{b}$.

PROOF Because A is invertible, we know that the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution, and we let $\mathbf{x}$ be this solution. Let X_k be the matrix obtained from the $n \times n$ identity matrix by replacing its kth-column vector by the column vector $\mathbf{x}$, so that

$$X_k = \begin{bmatrix} 1 & 0 & 0 & \cdots & x_1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & x_2 & 0 & 0 & & 0 \\ & & & & \vdots & & & & \\ 0 & 0 & 0 & \cdots & x_k & 0 & 0 & \cdots & 0 \\ & & & & \vdots & & & & \\ 0 & 0 & 0 & \cdots & x_n & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Let us compute the product AX_k. If $j \neq k$, then the jth column of AX_k is the product of A and the jth column of the identity matrix, which also yields the jth column of A. If $j = k$, then the jth column of AX_k is $A\mathbf{x} = \mathbf{b}$. Thus AX_k is the matrix obtained from A by replacing the kth column of A by the column vector $\mathbf{b}$. That is, AX_k is the matrix B_k described in the statement of the theorem. From the equation $AX_k = B_k$ and the multiplicative property of determinants, we obtain

$$\det(A) \cdot \det(X_k) = \det(B_k).$$

Computing $\det(X_k)$ by expanding by minors across the kth row, we see that $\det(X_k) = x_k$, and thus $\det(A) \cdot x_k = \det(B_k)$. Because A is invertible, we know that $\det(A) \neq 0$ and so $x_k = \det(B_k)/\det(A)$ as asserted in Equation (1). ▲

EXAMPLE 3 Solve the linear system

$$\begin{aligned} 5x_1 - 2x_2 + x_3 &= 1 \\ 3x_1 + 2x_2 \phantom{{}+x_3} &= 3 \\ x_1 + x_2 - x_3 &= 0, \end{aligned}$$

using Cramer's rule.

SOLUTION Using the notation in Theorem 4.5, we find that

$$\det(A) = \begin{vmatrix} 5 & -2 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & -1 \end{vmatrix} = -15, \qquad \det(B_1) = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 2 & 0 \\ 0 & 1 & -1 \end{vmatrix} = -5,$$

$$\det(B_2) = \begin{vmatrix} 5 & 1 & 1 \\ 3 & 3 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -15, \qquad \det(B_3) = \begin{vmatrix} 5 & -2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 0 \end{vmatrix} = -20.$$

Hence,

$$x_1 = \frac{-5}{-15} = \frac{1}{3},$$

$$x_2 = \frac{-15}{-15} = 1,$$

$$x_3 = \frac{-20}{-15} = \frac{4}{3}. \qquad \blacksquare$$

HISTORICAL NOTE CRAMER'S RULE appeared for the first time in full generality in the *Introduction to the Analysis of Algebraic Curves* (1750) by Gabriel Cramer (1704–1752). Cramer was interested in the problem of determining the equation of a plane curve of given degree passing through a certain number of given points. For example, the general second-degree curve, whose equation is

$$A + By + Cx + Dy^2 + Exy + x^2 = 0,$$

is determined by five points. To determine A, B, C, D, and E, given the five points, Cramer substituted the coordinates of each of the points into the equation for the second-degree curve and found five linear equations for the five unknown coefficients. Cramer then referred to the appendix of the work, in which he gave his general rule: "One finds the value of each unknown by forming n fractions of which the common denominator has as many terms as there are permutations of n things." He went on to explain exactly how one calculates these terms as products of certain coefficients of the n equations, how one determines the appropriate sign for each term, and how one determines the n numerators of the fractions by replacing certain coefficients in this calculation by the constant terms of the system.

Cramer did not, however, explain why his calculations work. An explanation of the rule for the cases $n = 2$ and $n = 3$ did appear, however, in *A Treatise of Algebra* by Colin Maclaurin (1698–1746). This work was probably written in the 1730s, but was not published until 1748, after his death. In it, Maclaurin derived Cramer's rule for the two-variable case by going through the standard elimination procedure. He then derived the three-variable version by solving two pairs of equations for one unknown and equating the results, thus reducing the problem to the two-variable case. Maclaurin then described the result for the four-variable case, but said nothing about any further generalization. Interestingly, Leonhard Euler, in his *Introduction of Algebra* of 1767, does not mention Cramer's rule at all in his section on solving systems of linear equations.

The most efficient way we have presented for computing a determinant is to row-reduce a matrix to triangular form. This is also the way we solve a square linear system. If A is a 10×10 invertible matrix, solving $A\mathbf{x} = \mathbf{b}$ using Cramer's rule involves row-reducing eleven 10×10 matrices $A, B_1, B_2, \ldots, B_{10}$ to triangular form. Solving the linear system by the method of Section 1.4 requires row-reducing just one 10×11 matrix so that the first ten columns are in upper-triangular form. This illustrates the folly of using Cramer's rule to solve linear systems. However, the structure of the components of the solution vector, as given by the Cramer's rule formula $x_k = \det(B_k)/\det(A)$, is of interest in the study of advanced calculus, for example.

The Adjoint Matrix

We conclude this section by finding a formula in terms of determinants for the inverse of an invertible $n \times n$ matrix $A = [a_{ij}]$. Recall the definition of the cofactor a'_{ij} from Eq. (2) of Section 4.2. Let $A_{i \to j}$ be the matrix obtained from A by replacing the jth row of A by the ith row. That is,

$$A_{i \to j} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{matrix} \\ \\ i\text{th row} \\ \\ j\text{th row} \\ \\ \\ \end{matrix}$$

Then

$$\det(A_{i \to j}) = \begin{cases} \det(A) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

If we expand $\det(A_{i \to j})$ by minors on the jth row, we have

$$\det(A_{i \to j}) = \sum_{s=1}^{n} a_{is} a'_{js},$$

and we obtain the important relation

$$\sum_{s=1}^{n} a_{is} a'_{js} = \begin{cases} \det(A) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2}$$

The term on the left-hand side in Eq. (2) is the entry in the ith row and jth column in the product $A(A')^T$, where $A' = [a'_{ij}]$ is the matrix whose entries are the cofactors of the entries of A. Thus Eq. (2) can be written in matrix form as

$$A(A')^T = (\det(A))I,$$

where I is the $n \times n$ identity matrix. Similarly, replacing the ith column of A by the jth column and by expanding on the ith column, we have

$$\sum_{r=1}^{n} a'_{ri}a_{rj} = \begin{cases} \det(A) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{3}$$

Relation (3) yields $(A')^T A = (\det(A))I$.

The matrix $(A')^T$ is called the **adjoint** of A and is denoted by $\text{adj}(A)$. We have established an important relationship between a matrix and its adjoint.

THEOREM 4.6 Property of the Adjoint

Let A be an $n \times n$ matrix. The adjoint $\text{adj}(A) = (A')^T$ of A satisfies

$$(\text{adj}(A))A = A(\text{adj}(A)) = (\det(A))I,$$

where I is the $n \times n$ identity matrix.

Theorem 4.6 provides a formula for the inverse of an invertible matrix, which we present as a corollary.

COROLLARY A Formula for the Inverse of an Invertible Matrix

Let $A = [a_{ij}]$ be an $n \times n$ matrix with $\det(A) \neq 0$. Then A is invertible, and

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A),$$

where $\text{adj}(A) = [a'_{ij}]^T$ is the transposed matrix of cofactors.

EXAMPLE 4 Find the inverse of

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

if the matrix is invertible, using the corollary of Theorem 4.6.

SOLUTION We find that $\det(A) = 4$, so A is invertible. The cofactors a'_{ij} are

$$a'_{11} = (-1)^2\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2, \qquad a'_{12} = (-1)^3\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = -2,$$

$$a'_{13} = (-1)^4\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = -4, \qquad a'_{21} = (-1)^3\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1,$$

$$a'_{22} = (-1)^4\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} = 1, \qquad a'_{23} = (-1)^5\begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix} = -4,$$

$$a'_{31} = (-1)^4\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = -2, \qquad a'_{32} = (-1)^5\begin{vmatrix} 4 & 1 \\ 2 & 0 \end{vmatrix} = 2$$

$$a'_{33} = (-1)^6\begin{vmatrix} 4 & 0 \\ 2 & 2 \end{vmatrix} = 8.$$

Hence,

$$A' = [a'_{ij}] = \begin{bmatrix} 2 & -2 & -4 \\ 1 & 1 & -4 \\ -2 & 2 & 8 \end{bmatrix}, \quad \text{so} \quad \operatorname{adj}(A) = \begin{bmatrix} 2 & 1 & -2 \\ -2 & 1 & 2 \\ -4 & -4 & 8 \end{bmatrix}$$

and

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{4}\begin{bmatrix} 2 & 1 & -2 \\ -2 & 1 & 2 \\ -4 & -4 & 8 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ -1 & -1 & 2 \end{bmatrix}.$$

▪

The method described in Section 1.5 for finding the inverse of an invertible matrix is more efficient than the method illustrated in the preceding example, especially if the matrix is large. The corollary is often used to find the inverse of a 2 × 2 matrix. We see that if $ad - bc \neq 0$, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

SUMMARY

1. A computationally feasible algorithm for finding the determinant of a matrix is to reduce the matrix to echelon form, using just row-addition and row-interchange operations. If a row of zeros is formed during the process, the determinant is zero. Otherwise, the determinant of the original matrix is found by computing $(-1)^r \cdot$ (Product of pivots) in the echelon form, where r is the number of row interchanges performed. This is one way to program a computer to find a determinant.
2. The determinant of a matrix can be found by row or column reduction of the matrix to a matrix having a sole nonzero entry in some column or row. One then expands by minors on that column or row, and continues this process. If a matrix having a zero row or column is encountered, the determinant is zero. Otherwise, one continues until the computation is reduced to the determinant of a 2 × 2 matrix. This is a good way to find a determinant when working with pencil and paper.
3. If A is invertible, the linear system $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x}$ whose kth component is given explicitly by the formula

$$x_k = \frac{\det(B_k)}{\det(A)},$$

where the matrix B_k is obtained from matrix A by replacing the kth column of A by $\mathbf{b}$.

4. The methods of Chapter 1 are far more efficient than those described in this section for actual computation of both the inverse of A and the solution of the system $A\mathbf{x} = \mathbf{b}$.
5. Let A be an $n \times n$ matrix, and let A' be its matrix of cofactors. The adjoint $\text{adj}(A)$ is the matrix $(A')^T$ and satisfies $(\text{adj}(A))A = A(\text{adj}(A)) = (\det(A))I$, where I is the $n \times n$ identity matrix.
6. The inverse of an invertible matrix A is given by the explicit formula

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A).$$

EXERCISES

In Exercises 1–10, find the determinant of the given matrix.

1. $\begin{bmatrix} 2 & 3 & -1 \\ 5 & -7 & 1 \\ -3 & 2 & -1 \end{bmatrix}$

2. $\begin{bmatrix} 4 & -3 & 2 \\ -1 & -1 & 1 \\ -5 & 5 & 7 \end{bmatrix}$

3. $\begin{bmatrix} 5 & 2 & 4 & 0 \\ 2 & -3 & -1 & 2 \\ 3 & -4 & 3 & 7 \\ 1 & -1 & 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 3 & -5 & -1 & 7 \\ 0 & 3 & 1 & -6 \\ 2 & -5 & -1 & 8 \\ -8 & 8 & 2 & -9 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 3 & -1 & 2 & 0 & 0 \\ 0 & 4 & 1 & -1 & 2 \\ 0 & 0 & -3 & 2 & 4 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ -1 & 4 & 1 & 0 & 0 \\ 0 & -3 & 5 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 2 & 0 & -3 \\ 0 & -2 & 1 & 0 & 0 \\ 5 & -3 & 2 & 0 & 0 \\ -3 & 4 & 0 & 0 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 2 & -1 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & -4 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 2 & -1 & 3 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 \\ -5 & 2 & 6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -2 & 8 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 2 & 1 \\ -1 & 2 & 4 & 0 & 0 \\ 3 & 1 & -2 & 0 & 0 \\ 5 & 1 & 5 & 0 & 0 \end{bmatrix}$

11. The matrices in Exercises 8 and 9 have zero entries except for entries in an $r \times r$ submatrix R and a separate $s \times s$ submatrix S whose main diagonals lie on the main diagonal of the whole $n \times n$ matrix, and where $r + s = n$. Prove that, if A is such a matrix with submatrices R and S, then $\det(A) = \det(R) \cdot \det(S)$.
12. The matrix A in Exercise 10 has a structure similar to that discussed in Exercise 11, except that the square submatrices R and S lie along the other diagonal. State and prove a result similar to that in Exercise 11 for such a matrix.
13. State and prove a generalization of the result in Exercise 11, when the matrix A has zero entries except for entries in k submatrices positioned along the diagonal.

In Exercises 14–19, use the corollary to Theorem 4.6 to find A^{-1} if A is invertible.

14. $A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

15. $A = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$

16. $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ -2 & 1 & 1 \end{bmatrix}$ 17. $A = \begin{bmatrix} 3 & 0 & 4 \\ -2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

18. $A = \begin{bmatrix} 3 & 0 & 3 \\ 4 & 1 & -2 \\ -5 & 1 & 4 \end{bmatrix}$ 19. $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}$

20. Find the adjoint of the matrix $\begin{bmatrix} 4 & 5 \\ -3 & 6 \end{bmatrix}$.

21. Find the adjoint of the matrix $\begin{bmatrix} 2 & 1 & 0 \\ 3 & 1 & 4 \\ 0 & 2 & 1 \end{bmatrix}$.

22. Given that $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det(A^{-1}) = 3$, find the matrix A.

23. If A is a matrix with integer entries and if $\det(A) = \pm 1$, prove that A^{-1} also has the same properties.

In Exercises 24–31, solve the given system of linear equations by Cramer's rule wherever it is possible.

24. $\begin{aligned} x_1 - 2x_2 &= 1 \\ 3x_1 + 4x_2 &= 3 \end{aligned}$ 25. $\begin{aligned} 2x_1 - 3x_2 &= 1 \\ -4x_1 + 6x_2 &= -2 \end{aligned}$

26. $\begin{aligned} 3x_1 + x_2 &= 5 \\ 2x_1 + x_2 &= 0 \end{aligned}$ 27. $\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + 2x_2 &= 2 \end{aligned}$

28. $\begin{aligned} 5x_1 - 2x_2 + x_3 &= 1 \\ x_2 + x_3 &= 0 \\ x_1 + 6x_2 - x_3 &= 4 \end{aligned}$

29. $\begin{aligned} x_1 + 2x_2 - x_3 &= -2 \\ 2x_1 + x_2 + x_3 &= 0 \\ 3x_1 - x_2 + 5x_3 &= 1 \end{aligned}$

30. $\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ x_1 + 2x_2 - x_3 &= 1 \\ x_1 - x_2 + 2x_3 &= 0 \end{aligned}$

31. $\begin{aligned} 3x_1 + 2x_2 - x_3 &= 1 \\ x_1 - 4x_2 + x_3 &= -2 \\ 5x_1 + 2x_2 &= 1 \end{aligned}$

In Exercises 32 and 33, find the component x_2 of the solution vector for the given linear system.

32. $\begin{aligned} x_1 + x_2 - 3x_3 + x_4 &= 1 \\ 2x_1 + x_2 + 2x_4 &= 0 \\ x_2 - 6x_3 - x_4 &= 5 \\ 3x_1 + x_2 + x_4 &= 1 \end{aligned}$

33. $\begin{aligned} 6x_1 + x_2 - x_3 &= 4 \\ x_1 - x_2 + 5x_4 &= -2 \\ -x_1 + 3x_2 + x_3 &= 2 \\ x_1 + x_2 - x_3 + 2x_4 &= 0 \end{aligned}$

34. Find the unique solution (assuming that it exists) of the system of equations represented by the partitioned matrix

$$\left[\begin{array}{cccc|c} a_1 & b_1 & c_1 & d_1 & 3b_1 \\ a_2 & b_2 & c_2 & d_2 & 3b_2 \\ a_3 & b_3 & c_3 & d_3 & 3b_3 \\ a_4 & b_4 & c_4 & d_4 & 3b_4 \end{array}\right].$$

35. Let A be a square matrix. Mark each of the following True or False.

___ a. The determinant of a square matrix is the product of the entries on its main diagonal.

___ b. The determinant of an upper-triangular square matrix is the product of the entries on its main diagonal.

___ c. The determinant of a lower-triangular square matrix is the product of the entries on its main diagonal.

___ d. A square matrix is nonsingular if and only if its determinant is positive.

___ e. The column vectors of an $n \times n$ matrix are independent if and only if the determinant of the matrix is nonzero.

___ f. A homogeneous square linear system has a nontrivial solution if and only if the determinant of its coefficient matrix is zero.

___ g. The product of a square matrix and its adjoint is the identity matrix.

___ h. The product of a square matrix and its adjoint is equal to some scalar times the identity matrix.

___ i. The transpose of the adjoint of A is the matrix of cofactors of A.

___ j. The formula $A^{-1} = (1/\det(A))\text{adj}(A)$ is of practical use in computing the inverse of a large nonsingular matrix.

36. Prove that the inverse of a nonsingular upper-triangular matrix is upper triangular.

37. Prove that a square matrix is invertible if and only if its adjoint is an invertible matrix.

38. Let A be an $n \times n$ matrix. Prove that $\det(\text{adj}(A)) = \det(A)^{n-1}$.

39. Let A be an invertible $n \times n$ matrix with $n > 1$. Using Exercises 37 and 38, prove that $\text{adj}(\text{adj}(A)) = (\det(A))^{n-2}A$.

The routine YUREDUCE in LINTEK has a menu option D that will compute and display the product of the diagonal elements of a square matrix. The routine MATCOMP has a menu option D to compute a determinant. Use YUREDUCE or MATLAB to compute the determinant of the matrices in Exercises 40–42. Write down your results. If you used YUREDUCE, use MATCOMP to compute the determinants of the same matrices again and compare the answers.

40. $\begin{bmatrix} 11 & -9 & 28 \\ 32 & -24 & 21 \\ 10 & 13 & -19 \end{bmatrix}$ 41. $\begin{bmatrix} 13 & -15 & 33 \\ -15 & 25 & 40 \\ 12 & -33 & 27 \end{bmatrix}$

42. $\begin{bmatrix} 7.6 & 2.8 & -3.9 & 19.3 & 25.0 \\ -33.2 & 11.4 & 13.2 & 22.4 & 18.3 \\ 21.4 & -32.1 & 45.7 & -8.9 & 12.5 \\ 17.4 & 11.0 & -6.8 & 20.3 & -35.1 \\ 22.7 & 11.9 & 33.2 & 2.5 & 7.8 \end{bmatrix}$

43. MATCOMP computes determinants in essentially the way described in this section. The matrix

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$$

has determinant 1, so every power of it should have determinant 1. Use MATCOMP with single-precision printing and with the default roundoff control ratio r. Start computing determinants of powers of A. Find the smallest positive integer m such that $\det(A^m) \neq 1$, according to MATCOMP. How bad is the error? What does MATCOMP give for $\det(A^{20})$? At what integer exponent does the break occur between the incorrect value 0 and incorrect values of large magnitude?

Repeat the above, taking zero for roundoff control ratio r. Try to explain why the results are different and what is happening in each case.

44. Using MATLAB, find the smallest positive integer m such that $\det(A^m) \neq 1$, according to MATLAB, for the matrix A in Exercise 43.

In Exercises 45–47, use MATCOMP in LINTEK or MATLAB and the corollary of Theorem 4.6 to find the matrix of cofactors of the given matrix.

45. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 0 \\ 3 & 1 & 4 \end{bmatrix}$

46. $\begin{bmatrix} -52 & 31 & 47 \\ 21 & -11 & 28 \\ 43 & -71 & 87 \end{bmatrix}$

47. $\begin{bmatrix} 6 & -3 & 2 & 14 \\ -3 & 7 & 8 & 1 \\ 4 & 9 & -5 & 3 \\ -8 & -40 & 47 & 29 \end{bmatrix}$

[HINT: Entries in the matrix of cofactors are integers. The cofactors of a matrix are continuous functions of its entries; that is, changing an entry by a very slight amount will change a cofactor only slightly. Change some entry just a bit to make the determinant nonzero.]

4.4 LINEAR TRANSFORMATIONS AND DETERMINANTS (OPTIONAL)

We continue our program of exhibiting the relationship between matrices and linear transformations. Associated with an $m \times n$ matrix A is the linear transformation T mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, where $T(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^n$. If $n = m$, so