Expansions by minors: Applications

The purpose here is to establish a few basic applications for expansions of determinants by minors. This material is not in Axler's book, but most of it is discussed more thoroughly in the file `week10/minors+cramer.pdf`, so most of the detailed derivations will be omitted.

We start by recalling the formula for expansion by minors along the i^{th} row of a matrix $A = (a_{i,j})$:

If $A_{i,j}$ is the matrix formed by deleting row i and column j from A, then

$$\det A \quad = \quad \sum_j \, (-1)^{i+j} \; a_{i,j} \det A_{i,j} \ .$$

If we now define the **classical adjoint** $\operatorname{adj} A$ (which is NOT the adjoint that was studied previously) to be the matrix whose i, j entry is $(-1)^{i+j} \, \det A_{j,i}$ (**note the transposition of indices!**), then the expansion by minors formula yields the following crucial identity:

$$A \cdot \operatorname{adj} A \quad = \quad \operatorname{adj} A \cdot A \quad = \quad \det A \cdot I$$

In particular, if A is an invertible matrix this yields the following formula for finding the inverse matrix:

$$A^{-1} \quad = \quad (\det A)^{-1} \cdot \operatorname{adj} A$$

And the latter yields a result generalizing the use of determinants for solving systems 2 or 3 linear equations in 2 or 3 unknowns from high school algebra.

THEOREM (Cramer's Rule; the name is pronounced kra-MARE)**.** *Let A be an $n \times n$ matrix, and let B be an $n \times 1$ matrix. If A is invertible (so that $\det A \neq 0$), then the unique solution to the matrix equation $AX = B$ is given by the formulas*

$$x_i \quad = \quad \frac{\det B[i]}{\det A}$$

where $B[i]$ is obtained from A by replacing the i^{th} column of A with B.■

Since it is much easier to find A^{-1} using row operations instead of determinants, the significance of Cramer's Rule and the identity for A^{-1} usually is more qualitative than computational. Here is an example to illustrate this principle:

THEOREM. *Let A be an invertible matrix such that each entry is an integer. Then A^{-1} is rational, and it is integral if and only if $\det A = \pm 1$.*

Proof. Since we know the entries of A are integers, the explicit formula for determinants shows that the entries of $\operatorname{adj} A$ are also integers, and also that $\det A$ is an integer. Therefore the determinant formula for A^{-1} implies that the latter's entries are all rational numbers. Furthermore, if $\det A = \pm 1$ then the entries of A^{-1} are integers.

Conversely, if A^{-1} has integer entries, then we also know that $\det A^{-1}$ is also an integer and therefore we have

$$1 \quad = \quad \det I \quad = \quad \det A \cdot \det A^{-1}$$

where all terms in the display are integers. Since the product of two integers is 1 if and only if both are equal to ± 1, it follows that $\det A = \pm 1$.■