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Mathematics 132, Winter 2004, Examination 2

Point values are indicated in brackets.

The first two problems are basically theoretical in nature, the last four basically computational.

1. [*15 points*] Recall that an orthogonal matrix is a square matrix A such that ${}^{\mathbf{T}}A\,A = I$. Prove that the determinant of an orthogonal matrix is equal to ± 1. [*Hint:* How are the determinants of A and its transpose related? Think about the 2×2 case.]

SOLUTION.

The determinats of A and ${}^{\mathbf{T}}A$ are equal, and therefore we have

$$1 \quad = \quad \det I \quad = \quad \det({}^{\mathbf{T}}A\,A) \quad = \quad \left(\det {}^{\mathbf{T}}A\right)\,(\det A) \quad = \quad (\det A)^2 \ .$$

Since the only real numbers x satsifying $x^2 = 1$ are ± 1, this means that $\det A$ must be equal to ± 1.■

2. [*20 points*] Recall that two square matrices A and B are similar if there is an invertible matrix P such that $B = P^{-1} A P$.

(i) Prove that if A is similar to B then A^2 is similar to B^2.

(ii) Prove that if B is invertible and A is a square matrix of the same size, then $A B$ is similar to $B A$.

SOLUTION.

(i) Since B is similar to A we have an invertible matrix P such that $B = P^{-1} A P$. Therefore we also have

$$B^2 \;=\; \left(P^{-1} A P\right) \cdot \left(P^{-1} A P\right) \;=\; P^{-1} A P P^{-1} A P \;=\; P^{-1} A I A P \;=\;$$

$$P^{-1} A A P \;=\; P^{-1} A^2 P$$

and hence B^2 is similar to A^2.■

(ii) This follows because

$$A B \;=\; I A B \;=\; \left(B^{-1} B\right) A B \;=\; B^{-1} (B A) B \;.■$$

3. [*15 points*] Recall that the inner product of two vectors with complex coordinates u_j and v_j is given by $\sum_j u_j \overline{v_j}$.

Find a complex number z such that the vectors $(i,\ 1-i,\ z)$ and $(1+i,\ 1-i,\ i)$ are orthogonal.

SOLUTION.

The inner product of these two vectors is equal to

$$[i \cdot (1-i)] \ + \ [(1-i)\cdot(1+i)] \ + \ [z \cdot (-i)] \ = \ [1+i] \ + \ 2 \ - \ z\,i \ = \ (3+i) \ - \ z\,i$$

and therefore the condition that the vectors are orthogonal reduces to the equation $(3+i) \ - \ z\,i = 0$. This means that

$$z \ = \ \frac{3+i}{i} \ = \ 1 \ - \ 3\,i \ . \blacksquare$$

4. [*15 points*] Find an orthogonal basis of eigenvectors for the following Hermitian matrix:

$$\begin{pmatrix} 1 & 2i \\ -2i & 1 \end{pmatrix}$$

SOLUTION.

The characteristic polynomial of the matrix is equal to $t^2 - 2t - 3 = (t-3)(t+1)$ and hence the eigenvalues are equal to 3 and -1. The eigenvectors corresponding to 3 are given by the null space of the matrix

$$\begin{pmatrix} -2 & 2i \\ -2i & 2 \end{pmatrix}$$

and thus the eigenvectors are the multiples of $(i, 1)$. If we multiply this vector by the reciprocal of its length, which is $\sqrt{2}$, we obtain a unit eigenvector for the eigenvalue 3.

Similarly, the eigenvectors corresponding to -1 are given by the null space of the matrix

$$\begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix}$$

and thus these eigenvectors are the multiples of $(-i, 1)$. If we multiply this vector by the reciprocal of its length, which is $\sqrt{2}$, we obtain a unit eigenvector for the eigenvalue -1.■

5. [*20 points*] The Jordan form for the matrix

$$\begin{pmatrix} 4 & 0 & 0 \\ 2 & 1 & 3 \\ 5 & 0 & 4 \end{pmatrix}$$

is given by one of the following two possibilities:

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad\qquad \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Determine which is the correct Jordan form and give reasons for your answer.

SOLUTION.

The first form corresponds to the case where there is a 2-dimensional space of eigenvectors for the eigenvalue 4, and the second corresponds to the case where the space of eigenvectors is not 2-dimensional (hence 1-dimensional). If A is the matrix given above then $A - 4I$ is equal to

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & -3 & 3 \\ 5 & 0 & 0 \end{pmatrix}$$

which has rank 2, and therefore the space of eigenvectors for the eigenvalue 4 is 1-dimensional. This means that the correct Jordan form is given by the second matrix in the display of possibilities.■

6. [*15 points*] Recall that a unitary matrix is one for which the columns, or equivalently the rows, form an orthonormal set.

Determine whether the matrix

$$\begin{pmatrix} (1+i)/\sqrt{3} & (1+i)/\sqrt{6} \\ -1/\sqrt{3} & a/\sqrt{6} \end{pmatrix}$$

is unitary for $a = 2$ or $a = 3$; the answer could be yes for both matrices, exactly one of the matrices or neither.

SOLUTION.

If $a = 3$ then the length of the second column is equal to the square root of $\frac{1}{6}(1^2 + 1^2 + 3^2) = 11/6$ and thus is not equal to 1. Therefore the matrix is not unitary if $a = 3$. On the other hand, if $a = 2$ then the length of the second column is equal to the square root of $\frac{1}{6}(1^2 + 1^2 + 2^2) = 1$. Also, the length of the first column is equal to $\frac{1}{3}(1^2 + 1^2 + 1^2) = 1$, so the matrix will be unitary if and only if the two columns are orthogonal. But the inner product of the first and second column is equal to

$$\frac{1+i}{\sqrt{3}} \cdot \frac{1-i}{\sqrt{6}} + \frac{-1}{\sqrt{3}} \cdot \frac{2}{\sqrt{6}} = \frac{2}{\sqrt{18}} + \frac{-2}{\sqrt{18}} = 0$$

and therefore the vectors in question are orthogonal, so that the matrix is unitary if $a = 2$.■