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Mathematics 132, Winter 2004, Examination 1

Point values are indicated in brackets.

1. [*30 points*] Show that the following matrix is diagonalizable, and find a nonzero eigenvector for the largest eigenvalue. Continue your work on the back side of this sheet or a clearly marked separate page if necessary.

$$A \quad = \quad \begin{pmatrix} 6 & 3 & -3 \\ -2 & 1 & 2 \\ 16 & 8 & -7 \end{pmatrix}$$

ADEQUATE PARTIAL SOLUTION.

It was enough to compute the characteristic polynomial and make some sort of effort to find the eigenvalues using this polynomial. Here is the characteristic polynomial calculation:

$$\begin{vmatrix} 6-t & 3 & -3 \\ -2 & 1-t & 2 \\ 16 & 8 & -7-t \end{vmatrix} \quad =$$

$$(6-t)\,(1-t)\,(-7-t) \;+\; 3{\cdot}2{\cdot}16 \;+\; (-3){\cdot}(-2){\cdot}8 \;-\; 8{\cdot}2{\cdot}(6-t) \;-\; 16{\cdot}(-3){\cdot}(1-t) \;-\; (-7-t){\cdot}(-2){\cdot}3 \quad =$$

$$(6-7\,t+t^2)\,(-7-t) \;+\; 3{\cdot}2{\cdot}16 \;+\; (-3){\cdot}(-2){\cdot}8 \;-\; 8{\cdot}2{\cdot}(6-t) \;-\; 16{\cdot}(-3){\cdot}(1-t) \;-\; (-7-t){\cdot}(-2){\cdot}3 \quad =$$

$$(-42+43\,t-t^3) \;+\; 96 \;+\; 48 \;-\; 16\cdot(6-t) \;+\; 48\cdot(1-t) \;+\; 6\cdot(-7-t) \quad =$$

$$(-42+43\,t-t^3) \;+\; 144 \;-\; 96 \;+\; 16\,t \;+\; 48 \;-\; 48\,t \;-\; 42 \;-\; 6\,t \quad =$$

$$(-42+144-96+48-42) \;+\; (43+16-48-6)\,t \;-\; t^3 \quad =$$

$$12 \;+\; 5\,t \;-\; t^3 \;. \blacksquare$$

Further discussion.

The characteristic polynomial turns out to factor as a product $(3-t)\cdot(4+3\,t+t^2)$ where the second factor has the two nonreal roots $\frac{1}{2}\,(1\pm\sqrt{-7})$. The eigenvectors for the real eigenvalue are just the solutions of the equation $(A-3I)\mathbf{x}=\mathbf{0}$, and these are given by the nonzero scalar multiples of

$$\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \;. \blacksquare$$

2. [*25 points*] Find an orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by $(1,1,0)$ and $(-1,2,1)$. Continue your work on the back side of this sheet or a clearly marked separate page if necessary.

SOLUTION.

Let the given vectors be $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. We begin by finding $\mathbf{u}_1$ using the formula $\mathbf{u}_1 = |\mathbf{v}_1|^{-1}\,\mathbf{v}_1$. Now $|\mathbf{v}_1|^2 = 2$ and therefore

$$\mathbf{u}_1 \quad = \quad \left(1/\sqrt{2},\, 1/\sqrt{2},\, 0\right)$$

completes the first step. The next step is to find a nonzero vector $\mathbf{w}$ that is a linear combination of $\mathbf{u}_1$ and $\mathbf{v}_2$ but is perpendicular to $\mathbf{u}_1$ using the formula

$$\mathbf{w} \quad = \quad \mathbf{v}_2 \; - \; \langle \mathbf{v}_2,\, \mathbf{u}_1\rangle\, \mathbf{u}_1$$

which in our case reduces to the following formula for $\mathbf{w}$:

$$\left(-1,\, 2,\, 1\right) \; - \; \left[(-1)\cdot(1/\sqrt{2} \; + \; (2)\cdot(1/\sqrt{2} \; + \; 1\cdot 0\right]\cdot \left(1/\sqrt{2},\, 1/\sqrt{2},\, 0\right)$$

If we simplify this we find that

$$\mathbf{w} \quad = \quad \left(-3,\, 3,\, 2\right)\,.$$

The vector $\mathbf{u}_2$ is just $\mathbf{w}$ divided by its length; the square of this length is equal to $9_9 + 4 = 22$, so therefore we have

$$\mathbf{u}_2 \quad = \quad \frac{1}{|\mathbf{w}|}\cdot\mathbf{w} \quad = \quad \frac{1}{\sqrt{22}}\cdot\left(-3,\, 3,\, 2\right)\,.$$

This completes the determination of an orthonormal basis for $\mathbf{u}_1$ and $\mathbf{u}_2$. Note that if we had decided to take $(-1,2,1)$ to be the first vector and $(1,1,0)$ to be the second, then we would have obtained a different orthonormal basis for the same subspace.∎

3. [*20 points*] Find the matrix for the orthogonal projection onto the subspace W of $\mathbb{R}^4$ spanned by the orthonormal vectors $(0, -\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ and $(\frac{2}{3}, 0, -\frac{1}{3}, \frac{2}{3})$.

SOLUTION.

If B is the 3×2 matrix whose columns are the given orthonormal vectors, then the matrix P for the projection is equal to the 3×3 matrix $B\,{}^T\!B$. Therefore the matrix we want is the product

$$\begin{pmatrix} -\frac{2}{3} & 0 \\ -\frac{2}{3} & \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} 0 & -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & 0 & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

and if we compute this matrix product we obtain the following value for P:

$$\frac{1}{9} \cdot \begin{pmatrix} -2 & 4 & 5 & 0 \end{pmatrix} \blacksquare$$

4. [*15 points*] Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{x} \in \mathbb{R}^n$, and let $\mathbf{x} = \widehat{\mathbf{x}} + \mathbf{x}'$ where $\widehat{\mathbf{x}} \in W$ and $\mathbf{x}'$ is perpendicular to every vector in W. Prove that the lengths of these vectors satisfy the relation

$$|\mathbf{x}|^2 \quad = \quad |\widehat{\mathbf{x}}|^2 \ + \ |\mathbf{x}'|^2 \ .$$

SOLUTION.

The square of the length of a vector is given by the formula $|\mathbf{x}|^2 \quad = \quad \langle \mathbf{x}, \, \mathbf{x} \rangle$ and if we apply this to the formula $\mathbf{x} = \widehat{\mathbf{x}} + \mathbf{x}'$ we obtain the following:

$$|\mathbf{x}|^2 \quad = \quad \langle \widehat{\mathbf{x}} + \mathbf{x}', \, \langle \widehat{\mathbf{x}} + \mathbf{x}' \rangle \quad =$$

$$\langle \widehat{\mathbf{x}}, \, \widehat{\mathbf{x}} \rangle \ + \ 2 \, \langle \widehat{\mathbf{x}}, \, \mathbf{x}' \rangle \ + \ \langle \mathbf{x}', \, \mathbf{x}' \rangle \ .$$

But since $\widehat{\mathbf{x}} \in W$ and $\mathbf{x}'$ is perpendicular to every vector in W the middle term is equal to zero, so we conclude that the expression under consideration is equal to

$$\langle \widehat{\mathbf{x}}, \, \widehat{\mathbf{x}} \rangle \ + \ \langle \mathbf{x}', \, \mathbf{x}' \rangle \quad = \quad |\widehat{\mathbf{x}}|^2 \ + \ |\mathbf{x}'|^2 \ .$$

5. [*10 points*] Give an example of a 2×2 matrix with real entries that is **NOT** diagonalizable and explain briefly why it is not diagonalizable.

SOLUTIONS.

A 2×2 matrix is diagonalizable if and only **either** if its characteristic polynomial has two distinct real roots **or else** the matrix is a diagonal matrix of the form λI, where λ is a real number (and hence the characteristic polynomial is $(\lambda - t)^2$).

Therefore a matrix that is not diagonalizable must meet one of the following criteria:

[1] The characteristic polynomial of the matrix has two nonreal roots. Specific examples of this are the matrices

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

where a and b are real numbers and $b \neq 0$ — the case $a = 0$ and $b = 1$ was given explicitly in the notes.

[2] The characteristic polynomial has the form $(\lambda - t)^2$ for some real number λ but the matrix is not equal to λI. Specific examples of this are the matrices

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

where λ is some real number — the case $\lambda = 2$ was given explicitly in the notes.■