The exponential function for matrices

Matrix exponentials provide a concise way of describing the solutions to systems of homogeneous linear differential equations that parallels the use of ordinary exponentials to solve simple differential equations of the form $y' = \lambda y$. For square matrices the exponential function can be defined by the same sort of infinite series used in calculus courses, but some work is needed in order to justify the construction of such an infinite sum. Therefore we begin with some material needed to prove that certain infinite sums of matrices can be defined in a mathematically sound manner and have reasonable properties.

Limits and infinite series of matrices

Limits of vector valued sequences in $\mathbf{R}^n$ can be defined and manipulated much like limits of scalar valued sequences, the key adjustment being that distances between real numbers that are expressed in the form $|s-t|$ are replaced by distances between vectors expressed in the form $|\mathbf{x}-\mathbf{y}|$. Similarly, one can talk about convergence of a vector valued infinite series $\sum_{n=0}^{\infty} \mathbf{v}_n$ in terms of the convergence of the sequence of partial sums $\mathbf{s}_n = \sum_{i=0}^{n} \mathbf{v}_k$. As in the case of ordinary infinite series, the best form of convergence is *absolute convergence*, which corresponds to the convergence of the real valued infinite series $\sum_{n=0}^{\infty} |\mathbf{v}_n|$ with nonnegative terms. A fundamental theorem states that a vector valued infinite series converges if the auxiliary series $\sum_{n=0}^{\infty} |\mathbf{v}_n|$ does, and there is a generalization of the standard M-test: *If $|\mathbf{v}_n| \leq M_n$ for all n where $\sum_n M_n$ converges, then $\sum_n \mathbf{v}_n$ also converges.*

We can view $m \times n$ matrices as mn-dimensional coordinate vectors, and we shall say that the *Euclidean magnitude* of a matrix is the usual length of the associated mn-dimensional vector.

In order to work with infinite series of matrices, we need some information about how the Euclidean magnitude behaves with respect to matrix products that is similar to the standard rule $|u\,v| = |u| \cdot |v|$ for absolute values. The following result provides the key estimate.

PRODUCT MAGNITUDE ESTIMATE. *Let A and B be matrices (not necessarily square) so that the product AB is defined, and for an arbitrary matrix C let $\|C\|$ be its Euclidean magnitude. Then $\|A\,B\| \leq \|A\| \cdot \|B\|$.*

Proof. It suffices to prove that the squares of the left and right hand sides are unequal in the same order. This is helpful because the squares of the Euclidean magnitudes are the sums of the squares of the matrix entries.

Given a matrix P let $\mathrm{Row}_i(P)$ and $\mathrm{Col}_j(P)$ denotes its $i*$th row and j^{th} column respectively. We then have

$$\|A\,B\|^2 \quad = \quad \sum_{i,j=1}^{n} \big(\mathrm{Row}_i(A) \cdot \mathrm{Col}_j(B)\big)^2$$

and applying the Schwarz inequality to each term in the sum we see that the latter is less than or equal to

$$\sum_{i,j=1}^{n} \left|\mathrm{Row}_i(A)\right|^2 \cdot \left|\mathrm{Col}_j(B)\right|^2 \quad = \quad \left(\sum_{i=1}^{n} \left|\mathrm{Row}_i(A)\right|^2\right) \cdot \left(\sum_{j=1}^{n} \left|\mathrm{Col}_j(B)\right|^2\right) .$$

But $\|A\|^2$ is equal to the first factor of this expression and $\|B\|^2$ is equal to the second.$\blacksquare$

One consequence of this estimate is the following matrix version of a simple identity for sums of infinite series:

INFINITE SUM FACTORIZATION. *Let $\sum_{k=1}^{\infty} A_k$ be a convergent infinite series of $m \times n$ matrices with sum S, and let P and Q be $s \times m$ and $n \times t$ matrices respectively. Then $\sum_{k=1}^{\infty} P\,A_k$ and $\sum_{k=1}^{\infty} A_k\,Q$ converge to $P\,S$ and $S\,Q$ respectively.*

Proof. Let S_r be the r^{th} partial sum of the original series. Then $P\,S_r$ and $S_r\,Q$ are the corresponding partial sums for the other two series, and we need to show that these two matrices become arbitrarily close to $P\,S$ and $S\,Q$ if r is sufficiently large. By the hypothesis we know the analogous statement is true for the original infinite series.

Let $\varepsilon > 0$ be given, and let L be the maximum of $\|P\| + 1$ and $\|Q\| + 1$. Choose R so large that $\|S_r - S\| < \varepsilon/L$ if $r \geq R$. It then follows that

$$\|P\,S_r - P\,S\| \quad \leq \quad \|P\| \cdot |\,S_r - S\| \quad < \quad \varepsilon$$

and similarly we have

$$\|S_r\,Q - S\,Q\| \quad \leq \quad \|Q\| \cdot |\,S_r - S\| \quad < \quad \varepsilon$$

so that the limits of the partial sums have their predicted values.■

Power series of matrices

in order to work with power series of matrices having the form

$$\sum_{k=0}^{\infty} c_k\,A^k$$

for suitable coefficients c_k, we need the following consequence of the Product Magnitude Estimate:

POWER MAGNITUDE ESTIMATE. *If A is a square matrix, then for all integers $k \geq 1$ we have $\|A^k\| \leq \|A\|^k$.*

Proof. This is a tautology if $k = 1$ so proceed by induction, assuming the result is true for $k - 1 \geq 1$. Then $A^k = A\,A^{k-1}$ and therefore by the preceding result and the induction hypothesis we have

$$\|A^k\| \quad = \quad \|A\,A^{k-1}\| \quad \leq \quad \|A\| \cdot \|A^{k-1}\| \quad \leq \quad |A\| \cdot \|A\|^{k-1} \quad = \quad \|A\|^k\ .■$$

COROLLARY. *Suppose that we are given a sequence of scalars c_k for which*

$$\lim_{k \to \infty} \frac{|c_{k+1}|}{|c_k|} \quad = \quad L$$

and A is a nonzero square matrix such that $\|A\|^{-1} > L$. Then the infinite matrix power series

$$\sum_{k=0}^{\infty} c_k\,A^k$$

converges absolutely.

Proof. The argument is closely related to the proof of the ratio test for ordinary infinite series. Upper estimates for the Euclidean magnitudes of the terms are given by the inequalities

$$\|c_k\, A^k\| \quad \le \quad |c_k| \cdot \|A\|^k$$

and the latter converges if $\|A\|^{-1} > L$ by the ratio test. But this means that the matrix power series converges absolutely.■

SPECIAL CASE. *If A is a square matrix, then the* **exponential series**

$$\exp(A) \quad = \quad \sum_{k=0}^{\infty} \frac{1}{k!}\, A^k$$

converges absolutely.■

Properties of matrix exponentials

It follows immediately that $\exp(0) = I$, and there is also a weak version of the usual law of exponents $e^{a+b} = e^a\, e^b$:

PRODUCT FORMULA. *If A and B are commuting matrices of the same size (i.e, $A\,B = B\,A$), then* $\exp(A+B) = \exp(A) \cdot \exp(B)$.

Idea of the proof. As for ordinary infinite series, one needs to do a little work in order to view the product of two infinite series sums as the sum of a third infinite series. Specifically, if one starts with convergent infinite series $\sum_k \; u_k$ and $\sum_k \; v_k$ with sums U and V, then one wants to say that $U\,V = \sum_k \; w_k$, where

$$w_k = \sum_{p+q=k} \; u_p \cdot v_q \; .$$

This turns out to be true if the original sequences are absolutely convergent, and one can use the same proof in the situation presented here because we know that A and B commute.■

It is important to note that the product formula does not necessarily hold if AB and BA are not equal.

COROLLARY. *For all square matrices A the exponential* $\exp(A)$ *is invertible and its inverse is* $\exp(-A)$.■

If the square matrix A is similar to a matrix B that has less complicated entries (for example, if A is similar to a diagonal matrix B), then the following result is often very helpful in understanding the behavior of $\exp(A)$.

SIMILARITY FORMULA. *Suppose that A and B are similar $n \times n$ matrices such that $B = P^{-1}\,A\,P$ for some invertible matrix P, then* $\exp(B) = P^{-1}\,\exp(A)\,P$.

Proof. By definition we have

$$P^{-1}\,\exp(A)\,P \quad = \quad P^{-1}\,\left(\sum_{k=0}^{\infty} \frac{1}{k!}\, A^k\right)\; P$$

and by the Infinite Sum Factorization formulas (proven above) and the identity $P^{-1}\,A^k\,P = (P^{-1}\,A\,P)^k$ we know that the right hand side is equal to

$$\sum_{k=0}^{\infty} \frac{1}{k!}\, P^{-1}\left(A^k\right)\,P \quad = \quad \sum_{k=0}^{\infty} \frac{1}{k!}\,\left(P^{-1}\,A\,P\right)^k \quad = \quad \sum_{k=0}^{\infty} \frac{1}{k!}\, B^k \quad = \quad \exp(B)$$

which is what we wanted to prove.■

Differentiating matrix valued functions

Differentiation of a matrix valued function of one real variable makes sense so long as the scalar valued functions corresponding to all the matrix entries are differentiable, and in this case one defines the derivative entry by entry. These derivatives have many familiar properties:

If C is a constant then $C' = 0$.

$(A + B)' = A' + B'$.

$(k\,A)' = k\,A' + k'\,A$.

$(A\,B)' = A'\,B + A\,B'$.

Perhaps the most noteworthy point is that one must watch the order of multiplication in the last of these identities.

Just as for ordinary power series, one has good term by term differentiation properties, and the proofs for ordinary infinite series go through with minimal changes:

TERMWISE DIFFERENTIATION. *Suppose that we have an infinite power series of $m \times n$ matrices $B(t) = \sum_{k=0}^{\infty} t^k\, B_k$ such that the radius of convergence for the auxiliary series $\beta(t) = \sum_{k=0}^{\infty} t^k\, \|B_k\|$ is at least r. Then the radius of convergence of $B(t)$ is at least r, and inside this radius of convergence we have $B'(t) = \sum_{k=0}^{\infty} t^k\,(k+1)\,B_{k+1}$.*■

If we apply this to the matrix exponential function $F(t) = \exp(tA)$ we obtain the equation

$$F'(t) \quad = \quad A\,\exp(tA) \quad = \quad \exp(tA)\,A\ .$$

All of this leads to the following result:

THEOREM. *For a given $n \times n$ matrix A and an $n \times 1$ column vector $\mathbf{b}$, there is a unique solution to the linear system of differential equations $X' = A\,X$ with initial condition $X(0) = \mathbf{b}$, and it is given by $\exp(tA)\,\mathbf{b}$.*

Proof. We first verify that the function described above is indeed a solution by applying the Leibniz rule. If $F(t) = \exp(tA)$, the latter says that the derivative of the function is given by the derivative of $F(t)\,\mathbf{b}$, which is equal to $F'(t)\,\mathbf{b}$, and by the discussion before the statement of the theorem this is equal to $A\,F(t)\,\mathbf{b}$. Also, the value at $t = 0$ is $\mathbf{b}$ because $\exp(0) = I$.

Conversely, suppose now that $X(t)$ solves the system of differential equation and has initial condition $X(0) = \mathbf{b}$. We proceed in analogy with the standard case where $n = 1$ and consider the product

$$W(t) \quad = \quad \exp(tA)^{-1}\,X(t) \quad = \quad exp(-tA)\,X(t)\ .$$

If we differentiate and apply the Leibniz Rule we obtain the following:

$$W'(t) \quad = \quad -A\,\exp(-tA)\,X(t) + \exp(-tA)\,X'(t) \quad = \quad -\exp(-tA)\,A\,X(t) + \exp(-tA)\,A\,X(t) \quad = \quad 0\ .$$

Therefore $W(t)$ is constant and equal to $W(0) = \mathbf{b}$. Left multiplication by $\exp(tA)$ then yields

$$\exp(tA)\,\mathbf{b} \quad = \quad \exp(tA)\,W(t) \quad = \quad \exp(tA)\,\exp(-tA)\,X(t) \quad = \quad I\,X(t) \quad = \quad X(t)\ .$$

This proves that the exponential solution is the only one with initial condition $X(0) = \mathbf{b}$.■