EXERCISES FOR MATHEMATICS 132

WINTER 2004

The references denote sections of the text for the course:

J. B. Fraleigh and R. A. Beauregard, *Linear Algebra* (Third Edition), Addison-Wesley, Reading MA *etc.*, 1994, ISBN 0–201–52675–1.

NOTE. Problems marked with two stars (**) are optional and particularly challenging; in particular, they are not part of the course material upon which examinations will be based.

I. Eigenvalues and eigenvectors

I.A : Review topics on matrices

(Fraleigh and Beauregard, §§1.5, 1.6, 4.2, 4.3)

Additional exercise

1. Suppose that A is an $n \times n$ matrix that is not invertible. Prove that there are nonzero $n \times n$ matrices B and C such that $AB = 0$ and $CA = 0$. [*Hint:* In the first case, find a nontrivial solution of the matrix equation $A\mathbf{x} = \mathbf{0}$ and look at a square matrix for which each column is equal to $\mathbf{x}$. Recall that a matrix A is invertible if and only if its transpose is invertible. Why does this mean that there is a nonzero $1 \times n$ row vector $\mathbf{y}$ such that $\mathbf{y}A = \mathbf{0}$?]

I.B : Review topics on linear transformations

(Fraleigh and Beauregard, §§2.3, 3.4, 4.4)

Additional exercises

1. Complete the verifications that the three examples at the end of Section I.B of the course notes define linear transformations, and prove the assertions in the first two examples about continuous functions being sent to continuous functions and infinitely differentiable functions being sent to infinitely differentiable functions.

2. The following problems deal with Examples **(1)** – **(3)** at the end of Section I.B in the course notes.

(a) In the third example at the end of Section I.B of the course notes, prove that Tf is a continuous function. [*Hints:* It is enough to show that on each square Q_L centered at the origin with length $2L$, formally defined by the inequalities

$$|u| \leq L, \qquad |v| \leq L$$

the quantity $|Tf(u) - Tf(v)| \leq C\,|u - v|$ for some positive constant C. Recall the inequality

$$\left| \int_0^1 h(y)\,dy \right| \leq \int_0^1 |h(y)|\,dy$$

and show that for all y we have

$$|k(u,y) - k(v,y)| \leq m(L)\,|u - v|$$

where $m(L)$ is the maximum value of the first partial derivative of k on the square Q_L. Use these observations to derive the sort of inequality given in the first sentence above.]

(b) Show that the image of T lies in the subspace $\mathcal{P}_n$ of polynomial functions of degree $\leq n$ if $k(x,y)$ is a polynomial of two variables of degree n.

(c) Show that Tf is infinitely differentiable. [*Hint:* Begin by showing that

$$[Tf]'(x) \quad = \quad \int_0^1 D_1 k(x,y)\,dy$$

where $D_1 k$ denotes the first partial derivative of k. How does this generalize?]

I.1 : Basic definitions

(Fraleigh and Beauregard, §5.1)

Fraleigh and Beauregard, § 5.1, p. 300: 2, 6, 10, 16, 22, 26, 36, 46*b*

Additional exercise

1. Suppose that A and B are $n \times n$ matrices such that $A\,B = B\,A$, and suppose that $\mathbf{v}$ is an eigenvector for A with associated eigenvector λ. Prove that $B\,\mathbf{v}$ is also an eigenvector for A with associated eigenvector λ.

I.2 : Diagonalization

(Fraleigh and Beauregard, §5.2)

Fraleigh and Beauregard, § 5.2, p. 315: 2, 4, 6, 10, 12, 14, 16, 22, 26

Additional exercise

1. Suppose that the $n \times n$ matrix A is diagonalizable.

(a) If b and c are scalars, prove that $C = A^2 + b\,A + I$ is also diagonalizable. [*Hint:* Make an educated guess about the eigenvectors of C.]

(b) If A is invertible, show that A^{-1} is also diagonalizable.

I.3 : Differential and difference equations

(Fraleigh and Beauregard, §5.3)

Fraleigh and Beauregard, § 1.7, p. 112: 2, 6, 10, 12, 14, 20, 24, 28, 32, 36, 38, 44, 48
Fraleigh and Beauregard, § 5.3, p. 325: 2, 4, 6, 8, 10, 12

Additional exercises

1. Calculate the monthly payments P_X for an amortized loan of \$200.000 with a 6 per cent annual interest rate (= a 0.5 per cent monthly interest rate) and repayment over X years, where $X = 15$, 20, 25 and 30 years, compute the fractions P_X/P_{15}, and compute the total payment ratios $T_X/T(15$ where $T_X = 12\,X\,P_X$.

2.** In a salt water desalinization plant there are two tanks of water. Suppose that tank 1 contains 1000 liters of brine in which 1000 kg. of salt is dissolved, and tank 2 contains 1000 liters of pure water. Suppose that water flows into tank 1 at the rate of 20 liters per minute and the mixture flows from tank 1 into tank 2 at a rate of 30 liters per minute. From tank 2, 10 liters is pumped back to tank 1 (establishing *feedback*) while 20 liters is flushed away. Find the amount of salt in both tanks at each time t. [*Hint:* Let $Y(t)$ denote the amount of salt in tanks 1 and 2 at time t, and find a system of first order linear equations satisfied by $Y(t)$.]

II. Perpendicularity (Orthogonality)

II.A : Review topics

(Fraleigh and Beauregard, §§ 1.2, 3.5)

Fraleigh and Beauregard, § 3.5, p. 236: 17, 23–26

Additional exercises

1. Prove that equality holds in the **weak** Schwarz inequality

$$\langle \mathbf{x}\,\mathbf{y} \rangle \quad \leq \quad |\mathbf{x}| \cdot |\mathbf{y}|$$

(no absolute value signs around the inner product) if and only if the vectors $\mathbf{x}$ and $\mathbf{y}$ are positive multiples of each other (assume both are nonzero to eliminate trivial cases). [*Hint:* Consider the quadratic function $f(t) = |\mathbf{x} - t\,\mathbf{y}|^2$, whose second degree term has a positive coefficient and whose values are always nonnegative. If we have a quadratic function $g(t) = a\,t^2 + b\,t + c$ that is always nonnegative with $a > 0$, what does this say about the discriminant $b^2 - 4\,a\,c$? If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent why is $f(t) > 0$ everywhere and why does this imply that one has strict inequality in the usual Schwarz inequality? If $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent with each a negative multiple of the other, explain why we have

$$\langle \mathbf{x}\,\mathbf{y} \rangle \quad = \quad -\,|\mathbf{x}| \cdot |\mathbf{y}| \ .$$

Finally, explain why there is equality in the weak Schwarz inequality if $\mathbf{x}$ and $\mathbf{y}$ are positive multiples of each other.]

2. * If V is an inner product space then one can define the *distance* between two vectors $\mathbf{x}$ and $\mathbf{y}$ in V is defined by the formula $d(\mathbf{x},\,\mathbf{y}) = |\mathbf{x} - \mathbf{y}|$.

(a) Show that the distance function has the following properties:

$d(\mathbf{x},\,\mathbf{y}) \geq 0$ with equality $\Longleftrightarrow$ we have $\mathbf{x} = \mathbf{y}$.

$d(\mathbf{x},\,\mathbf{y}) = d(\mathbf{y},\,\mathbf{x})$ for all $\mathbf{x}$ and $\mathbf{y}$.

$d(\mathbf{x},\,\mathbf{y}) \leq d(\mathbf{x},\,\mathbf{z}) + d(\mathbf{z},\,\mathbf{y})$ for all $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.

[*Hint:* The last one follows from the Triangle Inequality for vector length]

(b) Show that $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$ if and only if $\mathbf{z}$ is equal to $t\,\mathbf{x} + (1-t)\mathbf{y}$ for some $t \in [0,1]$. [*Hint:* By the Triangle and Schwarz Inequalities, one has equality in the distance relationship only if there is equality in the weak Schwarz Inequality for $\mathbf{z} - \mathbf{x}$ and $\mathbf{y} - \mathbf{z}$, and by the preceding exercise the latter holds only if each of these vectors is a multiple of the other. Show that this happens if and only if $\mathbf{z}$ can be written as a linear combination of $\mathbf{x}$ and $\mathbf{y}$ in the manner described above.]

II.1 : Orthogonal bases

(Fraleigh and Beauregard, §6.2)

Fraleigh and Beauregard, § 6.2, p. 347: 6, 8, 12, 18, 34

Additional exercises

1. Let V be an inner product space and let W be a finite dimensional subspace of V. Prove that $(W^{\perp})^{\perp} = W$.

2. Let $\mathcal{A} = \{\,\mathbf{a}_1, \;\cdots\;, \mathbf{a}_n\,\}$ be an ordered basis for $\mathbf{R}^n$.

(a) Prove that there is a unique *adjoint basis* for $\mathbf{R}^n$ of the form $\mathcal{A}^* = \{\,\mathbf{a}_1^*, \;\cdots\;, \mathbf{a}_n^*\,\}$ such that $\langle \mathbf{a}_i^*, \mathbf{a}_j\rangle$ is equal to 0 if $i \neq j$ and 1 if $i = j$. [*Hint:* Suppose that $\mathcal{A}$ and $\mathcal{B}$ are ordered bases and that A and B are matrices whose columns display the vectors of $\mathcal{A}$ and $\mathcal{B}$ in the correct order. Then the entries of ${}^{\mathbf{T}}B\,A$ are inner products of vectors in $\mathcal{B}$ with vectors in $\mathcal{A}$.]

(b) If $\mathcal{A}$ and $\mathcal{A}^*$ are as above and $\mathbf{x} \in \mathbf{R}^n$, prove that

$$\mathbf{x} = \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{a}_j^*\rangle\, \mathbf{a}_j \quad = \quad \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{a}_j\rangle\, \mathbf{a}_j^* \ .$$

$a_j = \langle \mathbf{a}.\, \mathbf{u}_j\rangle$.

3. An n**-dimensional flag** in $\mathbf{R}^n$ is defined to be a chain of subspaces

$$\mathcal{F} \quad = \quad W_1 \subset W_2 \subset \;\cdots\; \subset W_n = \mathbf{R}^n$$

such that $\dim W_i = i$ for all i. A *compatible orthonormal basis* for $\mathcal{F}$ is an ordered orthonormal basis $\mathcal{U} = \{\,\mathbf{u}_1, \;\cdots\;, \mathbf{u}_n\,\}$ such that each W_i is spanned by the first i vectors in $\mathcal{U}$.

(a) Suppose that we are given a sequence of vectors $\mathbf{a}_i$ such that $\mathbf{a}_1 \neq \mathbf{0}$ and $\mathbf{a}_i \notin W(i-1)$ for all $i \geq 2$. Prove that the first i vectors of the set $\mathcal{A} = \{\,\mathbf{a}_1, \;\cdots\;, \mathbf{a}_n\,\}$ form a basis for W_i.

(b) Prove that every n-dimensional flag has a compatible orthonormal basis $\mathcal{U}$ and if $\mathcal{V} = \{\,\mathbf{v}_1, \;\cdots\;, \mathbf{v}_n\,\}$ is any other compatible orthonormal basis then for all i we have $\mathbf{v}_i = \varepsilon_i\, \mathbf{u}_i$ where $\varepsilon_i = \pm 1$.

II.2 : Orthogonal projections and adjoints

(Fraleigh and Beauregard, §§6.1, 6.4)

Fraleigh and Beauregard, § 6.1, p. 336: 2, 6, 14*a*, 18, 20*abc*, 32, 34–36
Fraleigh and Beauregard, § 6.2, p. 347: 2, 4
Fraleigh and Beauregard, § 6.4, p. 368: 2, 10, 26

Additional exercises

1. Let V be a finite dimensional inner product space, and suppose that E_1 and E_2 are self-adjoint idempotents satisfying $E_1 E_2 = E_2 E_1$. Let W_1 and W_2 be the images of E_1 and E_2 respectively.

(a) Prove that $E_1 E_2$ and $E_1 + E_2 - E_1 E_2$ are the perpendicular projections onto $W_1 \cap W_2$ and $W_1 + W_2$ respectively.

(b) Prove that $E_1 E_2 = E_1$ if and only if $W_1 \subset W_2$.

2.* If V is a vector space, a linear transformation $T : V \to V$ is *idempotent* if and only if $T^2 = T$, and a linear transformation $S : W \to W$ is an *involution* if and only if $S^2 = I$.

(a) Given two linear transformations S and T from V to itself related by the equation $S = I - 2T$, prove that S is an involution if and only if T is idempotent.

(b) Suppose that S is an involution. Prove that every vector in V can be written uniquely in the form

$$\mathbf{v} \quad = \quad \mathbf{v}_+ \ + \ \mathbf{v}_-$$

where $\mathbf{v}_+$ and $\mathbf{v}_-$ are (zero or) eigenvectors for ± 1. [*Hints:* $\mathbf{v} \pm S(\mathbf{v})$ is zero or an eigenvector for ± 1. This is very similar to the argument proving that every continuous function on the real line is the sum of an even function and an odd one.]

(c) In the situation of the preceding exercises, prove that the map $T : V \to V$ sending $\mathbf{v}$ to $\mathbf{v}_-$ is an idempotent linear transformation such that $S = I - 2T$.

3. Suppose that V is a finite dimensional inner product space and $T : V \to V$ is a linear transformation; as usual, let T^* denote the adjoint of T.

(a) Prove that T is invertible if and only if T^* is invertible.

(b) Explain why the ranks of T and T^* are equal.

(c) Prove that the rank of T is is equal to the ranks of both $T\,T^*$ and $T^*\,T$ (hence all three ranks are equal).

4. Let V and W be finite dimensional inner product spaces, let $T : V \to W$ be a linear transformation, and let T^* be the adjoint of T.

(a) Prove that the kernel of T^* is the orthogonal complement of the image of T and the image of T^* is the orthogonal complement of the kernel of T.

(b) In analogy with page 138 of the text we shall define the *nullity* of T to be the dimension of the kernel of T, and as in the matrix case we have the formula $\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim V$. Prove that

$$\dim V - \dim W = \mathrm{nullity}(T) - \mathrm{nullity}(T^*) \ .$$

II.3 : Orthogonal matrices

(Fraleigh and Beauregard, §6.3)

Fraleigh and Beauregard, § 6.3, p. 358: 2, 4, 6, 10, 12, 20, 22, 28, 32, 34, 36

Additional exercise

1. Let A be an orthogonal 2×2 matrix.

(a) Using Exercise 39 on page 359 of the text, show that if $\det A = -1$ then A has $\pm, 1$ as eigenvalues and an orthonormal basis of eigenvectors. [*Hint:* First show that the characteristic polynomial is $t^2 - 1$. Why are eigenvectors for $+1$ and -1 perpendicular to each other?]

2.** Prove the Recognition Principle for orthogonal linear transformations that was stated in the notes: *Let V be a finite-dimensional inner product space, and let $T : V \to V$ be an zero-preserving isometric map — in other words, $T(\mathbf{0}) = \mathbf{0}$, every point of V has the form $T(\mathbf{x})$ for some $\mathbf{x}$, and $|\mathbf{x} - \mathbf{y}| = |T(\mathbf{x}) - T(\mathbf{y})|$ for all $\mathbf{x}, \mathbf{y} \in V$. Then T is an orthogonal linear transformation.* Here are the basic steps in the recommended approach:

(a) Show that T is 1–1. [*Hint:* Why is the distance between $T(\mathbf{x})$ and $T(\mathbf{y})$ positive if $\mathbf{x} \neq \mathbf{y}$?]

(b) Show that $|\mathbf{x}| = |T(\mathbf{x})|$ for all $\mathbf{x} \in V$ [*Hint:* We have $|\mathbf{x} - \mathbf{y}| = |T(\mathbf{x}) - T(\mathbf{y})|$ and $T(\mathbf{0}) = \mathbf{0}$).]

(c) Show that T preserves inner products, and in particular that T sends orthonormal sets into orthonormal sets, and likewise for orthonormal bases.

(d) Let $\mathcal{U}$ be an orthonormal basis for V, and let $\mathcal{U}'$ be its image under T. Express $\mathbf{x}$ and $T(\mathbf{x})$ as explicit linear combinations of the vectors in $\mathcal{U}$ and $\mathcal{U}'$ respectively using the inner product, and use the preceding step to show that T must be linear.

III. Change of bases

III.A : Review topics

(Fraleigh and Beauregard, §§ 2.3, 5.2)

Fraleigh and Beauregard, § 5.2, p. 315; 14, 16, 22

Additional exercises

1. For all scalars a and b the 3×3 matrix

$$A = \begin{pmatrix} 2 & a & 0 \\ 0 & -1 & b \\ 0 & 0 & +1 \end{pmatrix}$$

has a basis of eigenvectors. Find an invertible matrix P such that $P^{-1} A P$ is diagonal.

III.1 : Similarity of matrices

(Fraleigh and Beauregard, §§7.1–7.2)

Fraleigh and Beauregard, § 7.1, p. 394: 4, 10, 12
Fraleigh and Beauregard, § 7.2, p. 406: 2, 10, 26

Additional exercises

1. Let A and B be $n \times n$ matrices.

(a) If B is invertible, show that $A B$ is similar to $B A$.

(b) Give examples of noninvertible $n \times n$ matrices such that $A B$ and $B A$ are not similar. [*Hint:* Why is it enough to find 2×2 matrices A and B such that $A B = 0$ but $B A \neq 0$?]

2. Let A be an orthogonal 3×3 matrix.

(a) Prove that there is an orthogonal matrix P such that $P^{-1} A P$ has the form

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}$$

where the 2×2 matrix in the upper left hand corner is orthogonal and $\varepsilon = \pm 1$. [*Hint:* Find an ordered orthonormal basis such that the third vector $\mathbf{v}_3$ is an eigenvector for A and $\mathcal{L}_A$ takes the 2-dimensional subspace W of vectors perpendicular to $\mathbf{v}_3$ into itself.]

(b) If $\det A = 1$ does it follow that one can find a similar matrix of the above form with $\varepsilon = +1$? Either prove this or give a counterexample.

3. Suppose that we are given an invertible $n \times n$ matrix A and we write it as $Q\,R$ where Q is orthogonal and R is lower triangluar with positive entries down the diagonal. Why is $R\,Q$ similar to $Q\,R$? [*Note:* This simple observation is the basis for a very simple and effective algorithm that can be used to compute the eigenvalues of certain matrices. One can write $RQ = Q_1\,R_1$ by the **QR** decomposition theorem from the preceding unit, then one can write the similar matrix $R_1\,Q_1$ as $Q_2\,R_2$, and in favorable cases the resulting sequence of mutually similar matrices converges to a diagonal matrix. In cases that are not so favorable, one can sometimes modify the method slightly in order to obtain the desired eigenvalues.]

III.2 : Invariants of similarity

(Fraleigh and Beauregard, §§7.1–7.2)

[*The assignments from the text for* **III.1** *also cover this section*]

Additional exercises

1. Given an $n \times n$ matrix A with entries $a_{i,j}$, define its *trace*, denoted by $\mathrm{tr}(A)$, to be the scalar $\sum_i a_{i,i}$.

(a) Verify that the trace defines a linear transformation from the space of all $n \times n$ matrices to the real numbers (in the terminology of some books, such a map is also called a linear functional).

(b) Prove that $\mathrm{tr}(A\,B) = \mathrm{tr}(B\,A)$ for all $n \times n$, matrices A and B, and prove that similar matrices have the same trace.

(c) If A is diagonalizable, explain why its trace is equal to $\sum_\lambda \lambda\, n(\lambda)$, where λ runs through all the eigenvalues of A and $n(\lambda)$ is the dimension of the eigenspace for λ.

2. Explain why the matrix function $\varphi(A, B) = \mathrm{tr}\left({}^{\mathbf{T}}B\,A\right)$ defines an inner product on the space of $n \times n$ matrices.

IV. Complex linear algebra

IV.1 : Complex numbers

(Fraleigh and Beauregard, § 9.1)

Fraleigh and Beauregard, § 9.1, p. 463: 6, 10, 16, 22

Additional exercises

1. Using the formulas $e^{ix} = \cos x + i\,\sin x$ and $e^{i5y} = (e^{iy})^5$, derive the following trigonometric identities:

$$\cos 5x \;=\; \cos^5 x \;-\; 10\,\cos^3 x \cdot \sin^2 x \;+\; 5\,\cos x \cdot \sin^4 x$$

$$\sin 5x \;=\; \sin^5 x \;-\; 5\,\sin^3 x \cdot \cos^2 x \;+\; 10\,\sin x \cdot \cos^4 x$$

IV.2 : Complex matrices

(Fraleigh and Beauregard, §9.2)

Fraleigh and Beauregard, § 9.2, p. 472: 4, 10, 14*abc*, 20*abc*, 24, 26, 30*bd*, 36, 38

Additional exercises

1. Let χ be the map sending a vector

$$\mathbf{z} \;=\; (z_1, \;\cdots\;, z_n\,) \;\in\; \mathbb{C}^n$$

to the vector with conjugate entries:

$$\chi(\mathbf{z}) \;=\; (\overline{z_1}, \;\cdots\;, \overline{z_n}\,)$$

Give examples of subspaces W of the complex vector space $\mathbb{C}^n$ such that $\chi(W) = W$ and $\chi(W) \neq W$..

2. Let $\mathcal{A} = \{\,\mathbf{v}_1, \;\cdots\;, \mathbf{v}_k\,\}$ be a set of vectors in $\mathbb{C}^n$ with **real** coordinates. Prove that $\mathcal{A}$ is linearly independent in $\mathbb{C}^n$ if and only if it is linearly independent when viewed as a subset of $\mathbf{R}^n$. [*Hint:* Why is every vector in $\mathbb{C}^n$ uniquely expressible in the form $\mathbf{x} + i\,\mathbf{y}$ where both $\mathbf{x}$ and $\mathbf{y}$ have real coordinates?]

3. If V is a complex vector space, then one can also view V as a vector space over the real numbers by defining the real scalar product of $c \in \mathbf{R}$ and $\mathbf{v} \in V$ to be the complex scalar product $(c + 0\,i)\mathbf{v}$. Suppose that $W \subset \mathbb{C}^n$ is a complex subspace and $\mathcal{A} = \{\,\mathbf{v}_1, \;\cdots\;, \mathbf{v}_k\,\}$ is a basis for W as a complex vector space. Prove that $\mathcal{A} \cup i \cdot \mathcal{A}$ is a basis for W as a real vector space, where

$$i \cdot \mathcal{A} \;=\; \{\,i\,\mathbf{v}_1, \;\cdots\;, i\,\mathbf{v}_k\,\}\;.$$

What does this say about the relationship between the real and complex dimensions of W?

IV.3 : Complex eigenvalues and eigenvectors

(Fraleigh and Beauregard, §9.3)

Fraleigh and Beauregard, § 9.3, p. 485: 2, 6, 10, 12, 16, 24

Additional exercises

1. Let A be an $n \times n$ matrix over the complex numbers that is normal. Prove that A can be factored as the product of a unitary matrix U and a Hermitian matrix S with nonnegative real eigenvalues such that $S\,U = U\,S$. [*Hint:* Why is this true if A is diagonal? Think about what this means for 1×1 matrices. Use the Spectral Theorem to derive the general case.]

IV.4 : Jordan form

(Fraleigh and Beauregard, §9.4)

Fraleigh and Beauregard, § 9.4, p. 499: 8, 12, 18, 20 [only give the form for the last two]

Additional exercises

1. Suppose that $f(t)$ is a polynomial of degree n with rational coefficients whose highest degree term is equal to $(-1)^n\, t^n$. Show that there is an $n \times n$ matrix with rational entries whose characteristic polynomial is equal to $f(t)$. [*Hint:* Consider the matrix A whose j^{th} column is equal to the unit vector $\mathbf{e}_{j+1}$ for $j < n$ and whose last column is equal to

$$\sum_{k=0}^{n-1} (-1)^{n-1}\, c_k\, \mathbf{e}_{k+1}$$

and compute its characteristic polynomial using induction on the degree and expansion by minors.]

IV.5 : Differential equations revisited

(Fraleigh and Beauregard, §9.4)

Fraleigh and Beauregard, § 9.4, p. 499: 32c

Additional exercises

1. Let A be an $n \times n$ matrix with real entries that is orthogonal. Prove that there is a subspace $W \subset \mathbf{R}^n$ of dimension equal to 1 or 2 such that $\mathcal{L}_A$ maps vectors of W to vectors of W and also maps vectors of $W^{\perp}$ to vectors of $W^{\perp}$. [*Hint:* If A has a real eigenvalue, let W be the subspace spanned by a corresponding eigenvector. If A does not have a real eigenvalue, then it does have a complex eigenvalue λ with a corresponding eigenvector that can be written in the form $\mathbf{x} + i\,\mathbf{y}$ where both $\mathbf{x}$ and $\mathbf{y}$ have real coordinates. We also know that $\overline{\lambda}$ is an eigenvalue with eigenvector $\mathbf{x} - i\,\mathbf{y}$. Using these facts, show that $\mathcal{L}_A$ sends the real subspace spanned by $\mathbf{x}$ and $\mathbf{y}$ into itself.]

2.** Using mathematical induction and the result from the previous exercise, prove that if A is an orthogonal matrix then there is an orthogonal matrix P such that ${}^{\mathbf{T}}P\,A\,P$ is a block sum of orthogonal matrices where each summand is either 2×2 or 1×1.

3. Let A be a 2×2 matrix with real entries.

(a) If the characteristic polynomial is a product of linear factors over the real numbers, explain why A is similar to a matrix in Jordan form with real entries down the diagonal.

(b) If the characteristic polynomial is not a product of linear factors over the real numbers and $\det A > 0$, show that A is similar over the real numbers to a matrix of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

where the complex eigenvalues of A are $a \pm b\,i$.

4. Suppose that we are given a second order difference equation $X_{n+2} = 4\,X_{n+1} - 5\,X_n$. Find the general solution for this equation in terms of X_1 and X_0 over the complex numbers, and give a formula for the unique solution with $X_1 = X_0 = 1$ that does **NOT** involve complex numbers.

V. Quadratic forms

V.1 : Diagonalization of quadratic forms

(Fraleigh and Beauregard, §§ 6.3, 8.1)

Fraleigh and Beauregard, § 6.3, p. 358: 13–18
Fraleigh and Beauregard, § 8.1, p. 417: 6, 10, 16, 22

Additional exercises

1. Let A be a symmetric matrix with real entries.

(a) If the Rayleigh quotient is maximized at a given nonzero vector, why is it also maximized at every nonzero multiple of that vector?

(b) Can the Rayleigh quotient be maximized at two distinct nonzero vectors such that one is not a multiple of the other? Prove this or give a counterexample.

2. Let A be an $n \times n$ symmetric matrix, and let B be the symmetric $(n-1) \times (n-1)$ submatrix formed by deleting the last row and column.

(a) Using Rayleigh's principle, prove that the maximum eigenvalue of A is greater than or equal to the maximum eigenvalue of B and that the minimum eigenvalue of A less than or equal to the minimum eigenvalue of B.

(b) Under what condition is the maximum value for the Rayleigh quotient of B also the maximum value for the Rayleigh quotient of A?

(c) Apply the preceding considerations to the matrix A given below, finding a lower estimate for the maximum eigenvalue of A and showing that the maximum eigenvalue for B is strictly less than the maximum eigenvalue for A:

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(d) Both the sums and the products of the eigenvalues for A are positive integers. What are these integers?

V.2 : Classification of quadrics

(Fraleigh and Beauregard, §8.2)

Fraleigh and Beauregard, § 8.2, p. 429: 6, 12, 14, 16, 18, 20

V.3 : Classification of critical points

(Fraleigh and Beauregard, §8.3)

Fraleigh and Beauregard, § 8.3, p. 437: 2–14 even, 24–32 even

Additional exercises

1. Determine whether the following matrices are positive definite:

(a)

$$\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 9 & -1 & 2 \\ -1 & 7 & -3 \\ 2 & -3 & 3 \end{pmatrix}$$

(e)

$$\begin{pmatrix} 4 & -7 & -8 \\ -7 & 3 & -9 \\ -8 & -9 & 1 \end{pmatrix}$$

(f)

$$\begin{pmatrix} 6 & 7 & 1 \\ 7 & 9 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

(g)

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(h)

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

2. Let A be a symmetric matrix over the real numbers. Prove that there is a real number $c > 0$ such that $A + cI$ is positive definite.

3. Let A and B be a positive definite matrices. Prove that $A + B$ is positive definite. [*Note:* The Principal Minors Theorem is definitely **not** a good way to approach this result!]

4. Write the following positive definite matrix as a product ${}^{\mathbf{T}}B\,B$ for some invertible matrix B:

$$\begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}$$