SOLUTIONS TO SELECTED ADDITIONAL EXERCISES FOR MATHEMATICS 132 — Part 1

Winter 2004

I. Eigenvalues and eigenvectors

I.A : Review topic on matrices

1. Suppose that A is an $n \times n$ matrix that is not invertible. Prove that there are nonzero $n \times n$ matrices B and C such that $A\,B = 0$ and $C\,A = 0$. [*Hint:* In the first case, find a nontrivial solution of the matrix equation $A\,\mathbf{x} = \mathbf{0}$ and look at a square matrix for which each column is equal to $\mathbf{x}$. Recall that a matrix A is invertible if and only if its transpose is invertible. Why does this mean that there is a nonzero $1 \times n$ row vector $\mathbf{y}$ such that $\mathbf{y}\,A = \mathbf{0}$?]

SOLUTION.

The key point to notice is that if a matrix B is displayed in terms of columns as $B = (\mathbf{b}_1 \ \cdots \ \mathbf{b}_n)$ then we have

$$A\,B \quad = \quad \left(A\,\mathbf{b}_1 \ \cdots \ A\,\mathbf{b}_n\right) .$$

Therefore if we take $\mathbf{x} \neq \mathbf{0}$ so that $A\,\mathbf{x} = \mathbf{0}$ and let B be a matrix for which every column is equal to $\mathbf{x}$, then it will follow that $A\,B = 0$.

To prove the other statement, first observe that A is invertible if and only if its transpose is invertible, so if A is not invertible then there is a $\mathbf{z} \neq \mathbf{0}$ such that ${}^{\mathbf{T}}\!A\,\mathbf{z} = \mathbf{0}$. Let D be the matrix for which every column is equal to $\mathbf{z}$, so that ${}^{\mathbf{T}}\!A\,D = 0$ by the preceding paragraph. Taking transposes we have

$$0 \quad = \quad {}^{\mathbf{T}}0 \quad = \quad {}^{\mathbf{T}}\left({}^{\mathbf{T}}\!A\,D\right) \quad = \quad {}^{\mathbf{D}}\,A$$

and thus the nonzero matrix $C = {}^{\mathbf{T}}\!D$ satisfies $C\,A = 0$.■

I.1 : Basic definitions

1. Suppose that A and B are $n \times n$ matrices such that $A\,B = B\,A$, and suppose that $\mathbf{v}$ is an eigenvector for A with associated eigenvector λ. Prove that $B\,\mathbf{v}$ is also an eigenvector for A with associated eigenvector λ.

SOLUTION.

Since the problem asks for information on $A\,B\,\mathbf{v}$ we start by looking at this vector. Since $A\,B = B\,A$, the latter is equal to $B\,A\,\mathbf{v} = B(\lambda\,\mathbf{v}) = \lambda(B\,\mathbf{v})$, which is what we wanted to prove.■

I.2 : Diagonalization

1. Suppose that the $n \times n$ matrix A is diagonalizable.

(a) If b and c are scalars, prove that $C = A^2 + b\,A + I$ is also diagonalizable. [*Hint:* Make an educated guess about the eigenvectors of C.]

(b) If A is invertible, show that A^{-1} is also diagonalizable.

SOLUTION.

(a) Suppose that $\mathbf{v}$ is an eigenvector for A with associated eigenvalue λ. Then

$$C\,\mathbf{v} \;\; = \;\; \left[A^2 + b\,A + I\right]\,\mathbf{v} \;\; = \;\; A^2\,\mathbf{v} \; + \; b\,A\,\mathbf{v} \; + \; \mathbf{v} \;\; =$$

and since the discussion at the bottom of page 288 of the text the shows that $A^2\,|bfv = \lambda^2\,\mathbf{v}$, it follows that the right hand side of the displayed equation is equal to

$$\lambda^2\,\mathbf{v} \; + \; b\,\lambda\,\mathbf{v} \; + \; \mathbf{v} \;\; = \;\; \left(\lambda^2 \; + \; b\,\lambda \; + \; 1\right)\,\mathbf{v}$$

so that $\mathbf{v}$ is also an eigenvector for C. Therefore if we have a set of eigenvectors for A which is a basis for $\mathbf{R}^n$, this same basis also gives eigenvectors for C.■

SOLUTION.

(b) Suppose that $B = A^{-1}$ and $A\,\mathbf{v} = \lambda\,\mathbf{v}$ for some $\mathbf{v} \neq \mathbf{0}$. Since A is invertible it follows that $\lambda \neq 0$. We then have $\mathbf{v} = B(A\,\mathbf{v}) = B(\lambda\,\mathbf{v}) = \lambda(B\,\mathbf{v})$, which is equivalent to $B\,\mathbf{v} = \lambda^{-1}\,\mathbf{v}$. Therefore every eigenvector for A is an eigenvector for A^{-1}, and exactly as in the first part of the problem, if we have a set of eigenvectors for A which is a basis for $\mathbf{R}^n$, this same basis also gives eigenvectors for A^{-1}.■

I.3 : Differential and difference equations

1. Calculate the monthly payments P_X for an amortized loan of \$200.000 with a 6 per cent annual interest rate (= a 0.5 per cent monthly interest rate) and repayment over X years, where $X = 15$, 20, 25 and 30 years, compute the fractions P_X/P_{15}, and compute the total payment ratios $T_X/T(15$ where $T_X = 12\,X\,P_X$.

SOLUTION.

We start out with the formula from the notes:

$$P \;\; = \;\; \frac{r\,S\,(1+r)^N}{(1+r)^N - 1}$$

When using this formula in practice, it is important to remember that N represents the number of **months** and r corresponds to the monthly interest rate.

In this problem $r = 0.005$, $S = \$200{,}000$ and we want N to be either 180, 240, 300 or 360 depending upon whether X is equal to 15, 20, 25 or 30. If we substitute directly into the formula and round up to the next cent, we find that the payment values are as follows:

$P_{15} = \$1687.72$
$P_{20} = \$1432.87$
$P_{25} = \$1288.61$
$P_{30} = \$1119.11$

It follows that the monthly payment ratios P_X/P_{15} rounded off to three decimal places are equal to 0.849, 0.764 and 0.710 for $X = 20$, 25 and 30 respectively.

The total payments can be computed from the formula stated in the problem:

$T_{15} = \$303,788.60$
$T_{20} = \$343,888.80$
$T_{25} = \$363,583.00$
$T_{30} = \$431,679.60$

It follows that the total payment ratios T_X/T_{15} rounded off to three decimal places are equal to 1.132, 1.272 and 1.421 for $X = 20$, 25 and 30 respectively.■