SOLUTIONS TO SELECTED ADDITIONAL EXERCISES FOR MATHEMATICS 132 — Part 5

Winter 2004

V. Quadratic forms

V.1 : Diagonalization of quadratic forms

Problems from Fraleigh and Beauregard, § $6-3$, *pp.* $358-359$

14. We are asked to find the eigenvalues and orthonormal basis of eigenvectors for

$$A \;=\; \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix} .$$

The characteristic polynomial of A is $t^1 - 3\,t - 4$, which factors as $(4-t)\,(-1-t)$. Therefore the roots of the polynomial are 4 and -1. To find the eigenvector for -1 we must find the null space of

$$A \,+\, I \;=\; \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} .$$

This subspace is spanned by the vector $(1,-2)$. We could proceed to calculate the eigenvectors of A by finding the null space of $A - 4\,I$, but we can also use the fact that this eigenspace is orthogonal to the previous one to conclude that a vector that is orthogonal to $(1,-2)$ — for example, the vector $(2,1)$ — is an eigenvector for the eigenvalue 4.

To find an orthonormal basis of eigenvectors, we need to normalize these vectors; *i.e.*, multiply them by suitable nonzero constants so that the resulting products have length 1. The natural choice for the constant is the reciprocal of the length of a given vector. Both vectors in question have length equal to $\sqrt{5}$, and therefore the orthonormal basis we want is given by

$$1/\sqrt{5}\cdot\begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad 1/\sqrt{5}\cdot\begin{pmatrix} 2 \\ 1 \end{pmatrix} .$$

The desired matrix C is the one whose columns are these orthonormal eigenvectors.■

16. The first step is to compute the characteristic polynomial of the matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} .$$

Its characteristic polynomial is equal to $(-t)\,(3-t)\,(-1-t)$ and therefore its eigenvalues are 3, 0 and -1. The corresponding eigenspaces are the null spaces of the matrices

$$A-3I \;=\; \begin{pmatrix} -3 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -3 \end{pmatrix}, \qquad A-0I \;=\; \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad A+1I \;=\; \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and if one puts these matrices into their row reduced forms, the latter show that the corresponding eigenspaces are the multiples of the vectors

$$\mathbf{v}_3 \;=\; \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_0 \;=\; \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_{-1} \;=\; \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

We can divide these vectors by their lengths to get an orthonormal basis of eigenvectors. The lengths are given by $|\mathbf{v}_3| = \sqrt{6}$, $|\mathbf{v}_0| = \sqrt{3}$ and $|\mathbf{v}_{-1}| = \sqrt{2}$.■

18. The characteristic polynomial of the matrix is given by the 4×4 determinant

$$\begin{vmatrix} 1-t & -1 & 0 & 0 \\ -1 & 1-t & 0 & 0 \\ 0 & 0 & 1-t & 3 \\ 0 & 0 & 3 & 1-t \end{vmatrix}$$

and expansion by minors or other methods show that this determinant is equal to

$$(-t)\,(2-t)\,(-2-t)\,(4-t)\;.$$

The eigenvectors for these eigenvalues are given by

$$\mathbf{v}_0 \;=\; \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 \;=\; \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_{-2} \;=\; \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_4 \;=\; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

An orthonormal basis of eigenvectors is given by dividing each of these vectors by its length. In each case the length of the eigenvector in this dispaly is equal to $\sqrt{2}$.■

Problem from Fraleigh and Beauregard, § 8 − 1, p. 417

10. The first step is to find the symmetric matrix A such that the quadratic form can be written as ${}^{\mathbf{T}}\mathbf{v}\,A\,\mathbf{v}$. This matrix is given by

$$\begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}.$$

Its characteristic polynomial is equal to $t^2 - 3t - 4 = (t-4)\,(t+1)$. Its eigenvalues are therefore equal to 4 and -1, and the associated eigenvectors are $(1,-2)$ and $(2,1)$ as in Exercise 14 above. If we let P be the matrix whose columns are the associated unit eigenvectors then ${}^{\mathbf{T}}P\,A\,P$ is the diagonal matrix whose entries are -1 and 4. Therefore if we make the change of variables

$$\begin{pmatrix} x \\ y \end{pmatrix} \;=\; \frac{1}{\sqrt{5}} \cdot \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}$$

the original quadratic form will become $-u^2 + 4\,v^2$.■

Additional exercises

1. Let A be a symmetric matrix with real entries.

(a) If the Rayleigh quotient is maximized at a given nonzero vector, why is it also maximized at every nonzero multiple of that vector?

SOLUTION.

For all nonzero scalars c we have

$$r_A(c\,\mathbf{x}) \quad = \quad \frac{q_A(c\,\mathbf{x})}{|c\,\mathbf{x}|^2} \quad = \quad \frac{c^2\,q_A(\mathbf{x})}{c^2\,|\mathbf{x}|^2} \quad = \quad \frac{q_A(\mathbf{x})}{|\mathbf{x}|^2} \quad = \quad r_A(\mathbf{x}) \ .■$$

(b) Can the Rayleigh quotient be maximized at two distinct nonzero vectors such that one is not a multiple of the other? Prove this or give a counterexample.

SOLUTION.

The easiest example is to take the identity matrix, where the Rayleigh quotient is constant. Therefore the maximum is attained at every unit vector, and if A has more than one row and column this means there are many vectors for which the maximum value is attained such that none is a scalar multiple of the other.■

2. Let A be an $n \times n$ symmetric matrix, and let B be the symmetric $(n-1) \times (n-1)$ submatrix formed by deleting the last row and column.

(a) Using Rayleigh's principle, prove that the maximum eigenvalue of A is greater than or equal to the maximum eigenvalue of B and that the minimum eigenvalue of A less than or equal to the minimum eigenvalue of B.

SOLUTION.

Let $J : \mathbf{R}^2 \to \mathbf{R}^3$ be the standard inclusion map sending (u,v) to $(u,v,0)$. Then $\mathbf{x} \neq \mathbf{0}$ implies $J(\mathbf{x}) \neq \mathbf{0}$ and we also have

$$r_B(\mathbf{x}) \quad = \quad r_A\big(\,J(\mathbf{x})\,\big)$$

for all nonzero vectors $\mathbf{x} \in \mathbf{R}^2$. Therefore the maximum value of r_B is a value of the function r_A, and as such it must be less than or equal to the maximum value of r_A. Since the maximum values of r_B and r_A are the maximum eigenvalues of B and A respectively, it follows that the maximum eigenvalue of B is less than or equal to the maximum eigenvalue of A.

Similarly, it follows that the minimum value of r_B is a value of the function r_A, and as such it must be greater than or equal to the minimum value of r_A. Since the minimum values of r_B and r_A are the minimum eigenvalues of B and A respectively, it follows that the minimum eigenvalue of B is greater than or equal to the minimum eigenvalue of A.■

(b) Under what condition is the maximum value for the Rayleigh quotient of B also the maximum value for the Rayleigh quotient of A?

SOLUTION.

The maximum value of a Rayleigh quotient is an eigenvalue, so if the maximum values are equal it follows that the common maximum must be an eigenvalue for both A and B.■

(c) Apply the preceding considerations to the matrix A given below, finding a lower estimate for the maximum eigenvalue of A and showing that the maximum eigenvalue for B is strictly less than the maximum eigenvalue for A:

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

SOLUTION.

As in the preceding part of this exercises, let B be the 2×2 matrix in the upper left hand corner. Then direct computation shows that $\chi_B(t) = t^2 - 5t + 2$, and its roots are $\frac{1}{2}(5 \pm \sqrt{17})$. Therefore the maximum eigenvalue of A is at least as large as the larger of these two roots, which is $\frac{1}{2}(5 + \sqrt{17}) \approx 4.5615528$.

To see that this is strictly less than the maximum eigenvector for A, it suffices to find an eigenvector $\mathbf{x}$ for the maximum eigenvalue of B and show that $J(\mathbf{x})$ is not an eigenvector for A (for then the maximum of the Rayleigh quotient must be greater than $r_B(\mathbf{x})$, which is the maximum eigenvalue for B). The first step in this argument is to find an eigenvector for $\frac{1}{2}(5 + \sqrt{17})$. Direct calculation shows that one is given by

$$\mathbf{x} \quad = \quad \begin{pmatrix} \frac{1}{2}(5 + \sqrt{17}) \\ 2 \end{pmatrix}$$

and the corresponding vector $J(\mathbf{x})$ is not an eigenvector for A, one reason being that the third coordinate of $A[J(\mathbf{x})]$ is nonzero.■

(d) Both the sums and the products of the eigenvalues for A are positive integers. What are these integers?

SOLUTION.

The sum is the trace of the matrix and the product is its determinant.■

V.2 : Classification of quadrics

Problems from Fraleigh and Beauregard, § 8 − 2, pp. 429 − 430

12. The first step is to find the symmetric coefficient matrix for the quadratic form, which is equal to

$$A \quad = \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 10 \end{pmatrix} .$$

The characteristic polynomial of this matrix is equal to $(2-t)(t^2 - 12t + 11) = (2-t)(11-t)(1-t)$. Therefore the eigenvalues are 1, 2 and 11. This means that there is an orthogonal change of variables that transforms the original equation into $u_1^2 + 2u_2^2 + 11u_3^2 = 9$. This is the equation of an ellipsoid.■

14. The fucntion in this case has a first degree term. If it is possible to make a change of coordinates that will eliminate all such terms right away, one should do so. In this case it is possible by completing the square for the second variable, which is formally given by the change of variables

$X = x$, $Z = z$ and $Y = y + 1$. The set in question is then defined by the equation $2\,X\,Z + Y^2 = 0$. The symmetric matrix for the quadratic form on the left hand side of this equation is given by

$$A \;=\; \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and its characteristic polynomial is equal to $(1-t)^2\,(-1-t)$, so that it has a 2-dimensional eigenspace for the eigenvalue $+1$ and a 1-dimensional eigenspace for the eigenvalue -1. This means that there is an orthogonal change of variables that transforms the original equation into $u_1^2 + u_2^2 - u_3^2 = 0$. This is the equation of a cone.∎

16. The the symmetric coefficient matrix for the quadratic form is equal to

$$A \;=\; \begin{pmatrix} 3 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 3 \end{pmatrix}$$

and the characteristic polynomial of this matrix is equal to $(-t)\,(2-t)\,(6-t)$. Therefore the eigenvalues are 0, 2 and 6. This means that there is an orthogonal change of variables that transforms the original equation into $2\,u_2^2 + 6\,u_3^2 = 1$. This is the equation of an elliptic cylinder.∎

V.3 : Classification of critical points

Problems from Fraleigh and Beauregard, § 8 − 3, pp. 437 − 438

2. The associated quadratic form $-2\,x^2 + 8\,x\,y - 3\,y^2$ has symmetric coefficient matrix

$$\begin{pmatrix} -2 & 4 \\ 4 & -3 \end{pmatrix}$$

and the principal minors of this matrix are -2 and -10. Since the determinant itself is negative, this means that one has a saddle point, and hence no local extremum at $(0,0)$.∎

4. The associated quadratic form $-8\,x^2 + 6\,x\,y - 2\,y^2$ has symmetric coefficient matrix

$$\begin{pmatrix} -8 & 3 \\ 3 & -2 \end{pmatrix}$$

and the principal minors of this matrix are -8 and 7. Since the determinant itself is positive, this means that one has either a relative maximum or a relative minimum, and since the first principal minor is negative it follows that one has a realtive maximum at $(0,0)$.∎

6. The associated quadratic form $8\,x^2 + 4\,x\,y + y^2$ has symmetric coefficient matrix

$$\begin{pmatrix} 8 & 2 \\ 2 & 1 \end{pmatrix}$$

and the principal minors of this matrix are 8 and 4. Since the determinant itself is positive, this means that one has either a relative maximum or a relative minimum, and since the first principal minor is positive it follows that one has a realtive minimum at $(0,0)$.∎

12. The associated quadratic form $2x^2 + 6xz - y^2 + 5z^2$ has symmetric coefficient matrix

$$\begin{pmatrix} 2 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & 5 \end{pmatrix}$$

and the principal minors of this matrix are 2, -2 and 1. If we restrict the function to the set of points where $z = 0$, the second condition implies that the restricted function does not have a relative maximum or minimum at $(0,0)$. But if the restricted function does not have a relative maximum or minimum at $(0,0)$, then certainly the original function also cannot have a relative maximum or minimum at $(0,0,0)$.■

Additional exercises

1. Determine whether the following matrices are positive definite:

(a)

$$\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$$

SOLUTION.

The upper left entry is equal to the positive number 2 and the determinant is equal to 1. Therefore the matrix is positive definite by the Principal Minors Test.■

(b)

$$\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$$

SOLUTION.

The upper left entry is equal to the positive number 1 and the determinant is equal to -1. Therefore the matrix is NOT positive definite by the Principal Minors Test.■

(c)

$$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

SOLUTION.

The entry in the upper left position is the positive number 4, the determinant of the 2×2 matrix in the upper left hand corner is 12, and the determinant of the entire matrix is

$$64 + 8 + 8 - 16 - 16 - 16 = 32 > 0$$

and therefore the matrix is positive definite by the Principal Minors Test.■

(d)

$$\begin{pmatrix} 9 & -1 & 2 \\ -1 & 7 & -3 \\ 2 & -3 & 3 \end{pmatrix}$$

SOLUTION.

The entry in the upper left position is the positive number 9, the determinant of the 2×2 matrix in the upper left hand corner is 64, and the determinant of the entire matrix is

$$(63 \times 3) \ + \ 6 \ + \ 6 \ - \ 28 \ - \ 81 \ - \ 3 \ = \ 89 \ > \ 0$$

and therefore the matrix is positive definite by the Principal Minors Test.∎

(e)

$$\begin{pmatrix} 4 & -7 & -8 \\ -7 & 3 & -9 \\ -8 & -9 & 1 \end{pmatrix}$$

SOLUTION.

The element in the upper right hand corner is 4, and the determinant of the 2×2 matrix in the upper right hand corner is $12 - 49 = -37 < 0$, so the matrix is NOT positive definite by the Principal Minors Test.∎

(f)

$$\begin{pmatrix} 6 & 7 & 1 \\ 7 & 9 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

SOLUTION.

The element in the upper right hand corner is 7, the determinant of the 2×2 matrix in the upper right hand corner is $54 - 49 = 5 > 0$, and the determinant of the entire matrix is

$$54 \ + \ 14 \ + \ 14 \ - \ 9 \ - \ 24 \ - \ 49 \ = \ 82 \ - \ 33 \ - \ 49 \ = \ 0$$

so the matrix is NOT positive definite by the Principal Minors Test.∎

(g)

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

SOLUTION.

The first and third rows of the matrix are equal, so its determinant is zero, and therefore the matrix NOT positive definite by the Principal Minors Test.∎

(h)

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

SOLUTION.

The entry in the upper left position is the positive number 2, the determinant of the 2×2 matrix in the upper left hand corner is 3, and the determinant of the 3×3 matrix in the upper left hand corner is $8 - 2 - 2 = 4 > 0$, and a computation using expansion by minors along the first row shows that the determinant of the entire matrix is equal to

$$2 \cdot \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} \;-\; (-1) \cdot \begin{vmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} \;=\; 2 \cdot 4 + \big(-4 - (-1)\big) \;=\; 5 \;>\; 0$$

and therefore the matrix is positive definite by the Principal Minors Test.∎

2. Let A be a symmetric matrix over the real numbers. Prove that there is a real number $c > 0$ such that $A + cI$ is positive definite.

SOLUTION.

Let $c = |\lambda_1| + 1$ where λ_1 is the minimum eigenvalue. Then the eigenvalues of $A + c\,I$ are given by

$$\lambda_j + c \;\geq\; \lambda_1 + c \;=\; \lambda_1 + |\lambda_1| + 1 \;\geq\; 1 \;>\; 0$$

(since $\lambda + |\lambda| \geq \lambda - \lambda \geq 0$) and since the eigenvalues of $A + c\,I$ are all positive it follows that the latter is positive definite.∎

3. Let A and B be a positive definite matrices. Prove that $A + B$ is positive definite. [*Note:* The Principal Minors Theorem is definitely **not** a good way to approach this result!]

SOLUTION.

By the positive definiteness of A and B we know that ${}^{\mathbf{T}}\mathbf{x}\,A\,\mathbf{x} > 0$ and ${}^{\mathbf{T}}\mathbf{x}\,B\,\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$. Adding these inequalities, we see that

$${}^{\mathbf{T}}\mathbf{x}\,A\,\mathbf{x} \;+\; {}^{\mathbf{T}}\mathbf{x}\,B\,\mathbf{x} \;=\; {}^{\mathbf{T}}\mathbf{x}\,\big(A\,\mathbf{x} + B\,\mathbf{x}\big) \;=\; {}^{\mathbf{T}}\mathbf{x}\,\big(\,(A+B\,)\,\mathbf{x}\big) \;=\; {}^{\mathbf{T}}\mathbf{x}\,(A+B)\,\mathbf{x}$$

is also positive for all $\mathbf{x} \neq \mathbf{0}$.∎

4. Write the following positive definite matrix as a product ${}^{\mathbf{T}}B\,B$ for some invertible matrix B:

$$\begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}$$

SOLUTION.

There are several ways to do this. Perhaps the easiest is to look for a symmetric square root of the matrix A displayed above. One way of constructing such a square root is by diagonalization. We have $\chi_A(t) = t^2 - 13\,t + 36$, which factors as $(9 - t)\,(4 - t)$. Eigenvectors for 4 and 9 are given by

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

respectively. Therefore if P is the orthogonal matrix

$$\frac{1}{\sqrt{5}} \cdot \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

then ${}^{\mathbf{T}}P\,A\,P$ is the matrix D whose entries are 4 and 9 respectively. This matrix has an obvious square root given by the diagonal matrix C whose entries are 2 and 3, and in fact the matrix

$$B \;=\; P\,C\,{}^{\mathbf{T}}P$$

is symmetric and satisfies $B^2 = A$. For the sake of completeness, here is the proof: We have

$$B^2 \;=\; \left(P\,C\,{}^{\mathbf{T}}P\right)^2 \;=\; \left(P\,C\,{}^{\mathbf{T}}P\right)\cdot\left(P\,C\,{}^{\mathbf{T}}P\right) \;=\; P\,C\,{}^{\mathbf{T}}P\,P\,C\,{}^{\mathbf{T}}P$$

and since P is orthogonal we know that ${}^{\mathbf{T}}P\,P$ is the identity matrix so that the right hand side reduces to

$$P\,C\,I\,C\,{}^{\mathbf{T}}P \;=\; P\,C^2\,{}^{\mathbf{T}}P \;=\; P\,D\,{}^{\mathbf{T}}P \;=\; P\left({}^{\mathbf{T}}P\,A\,P\right){}^{\mathbf{T}}P \;=\; P\,{}^{\mathbf{T}}P\,A\,P\,{}^{\mathbf{T}}P\,.$$

Since P is orthogonal we may simplify this further to $I\,A\,I$, which is equal to A. Therefore B will be a symmetric invertible square root of A and hence the latter will be equal to ${}^{\mathbf{T}}B\,B$.

All that remains now is to compute B, which is equal to

$$\frac{1}{\sqrt{5}}\cdot\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}\cdot\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\cdot\frac{1}{\sqrt{5}}\cdot\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}$$

and if we carry out all the multiplications we obtain the following result:

$$B \;=\; \frac{1}{5}\cdot\begin{pmatrix}14 & -2\\ -2 & 11\end{pmatrix}$$

One can (and should!) verify by direct computation that $B^2 = A$.$\blacksquare$