SOLUTIONS TO SELECTED ADDITIONAL EXERCISES FOR MATHEMATICS 132 — Part 2

Winter 2004

I. Eigenvalues and eigenvectors

I.3 : Differential and difference equations

Problems from Fraleigh and Beauregard, § 1 − 7, pp. 112 − 114

2. Yes. The matrix is regular because the square of the matrix has no zero entries.

6. Yes. The matrix is not regular because every power has the same third column, and the latter has zero entries. power

10. One computes the $(2,3)$ entry of T^2, and the value is 0.23.

20. One needs to find an eigenvector for the matrix associated to the eigenvalue 1 such that the sums of the coordinates are nonnegative and add up to 1. The eigenvectors associated to this eigenvalue are the vectors having the form

$$\begin{bmatrix} 2\,r \\ r \end{bmatrix}$$

where r is an arbitrary real number, and if we normalize so that the sum of the coordinates is equal to 1 we obtain the desired vector, which is

$$\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

48.

$$\begin{bmatrix} 5/18 \\ 1/3 \\ 7/18 \end{bmatrix}$$

Problem from Fraleigh and Beauregard, § 5 − 3, p. 325

8. The general solution of the system is given by

$$k_1\, e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \;+\; k_2\, e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where k_1 and k_2 are arbitrary real numbers ("constants of integration").

II. Perpendicularity (Orthogonality)

II.A : Review topics

Problems from Fraleigh and Beauregard, § 3 − 5, pp. 236 − 237

26. Let V be an inner product space, and let $\mathbf{v}, \mathbf{w} \in V$. Prove that the vectors

$$|\mathbf{v}| \cdot \mathbf{w} \ \pm\ |\mathbf{w}| \cdot \mathbf{v}$$

are perpendicular.

SOLUTION.

We need to verify that the inner product of the two vectors is equal to zero, so the idea is to expand the inner product and look for various terms to cancel. Here is how it works. We have

$$\Big\langle |\mathbf{v}|\ \mathbf{w} \ +\ |\mathbf{w}|\ \mathbf{v},\ |\mathbf{v}|\ \mathbf{w} \ -\ |\mathbf{w}|\ \mathbf{v} \Big\rangle \ =$$

$$|\mathbf{v}|^2 \langle \mathbf{w}, \mathbf{w} \rangle \ +\ |\mathbf{v}|\,|\mathbf{w}|\,\langle \mathbf{v}, \mathbf{w} \rangle \ -\ |\mathbf{v}|\,|\mathbf{w}|\,\langle \mathbf{w}, \mathbf{v} \rangle \ -\ |\mathbf{w}|^2 \langle \mathbf{v}, \mathbf{v} \rangle$$

and in this expression the middle two terms cancel because the inner product is commutative. Thus we are left with

$$|\mathbf{v}|^2 \langle \mathbf{w}, \mathbf{w} \rangle \ -\ |\mathbf{w}|^2 \langle \mathbf{v}, \mathbf{v} \rangle \ = \ |\mathbf{v}|^2\,|\mathbf{w}|^2 \ -\ |\mathbf{w}|^2\,|\mathbf{v}|^2 \ = \ 0$$

which is what we were asked to prove.∎

Additional exercise

1. Prove that equality holds in the **weak** Schwarz inequality

$$\langle \mathbf{x}\,\mathbf{y} \rangle \ \leq\ |\mathbf{x}| \cdot |\mathbf{y}|$$

(no absolute value signs around the inner product) if and only if the vectors $\mathbf{x}$ and $\mathbf{y}$ are positive multiples of each other (assume both are nonzero to eliminate trivial cases). [*Hint:* Consider the quadratic function $f(t) = |\mathbf{x} - t\,\mathbf{y}|^2$, whose second degree term has a positive coefficient and whose values are always nonnegative. If we have a quadratic function $g(t) = a\,t^2 + b\,t + c$ that is always nonnegative with $a > 0$, what does this say about the discriminant $b^2 - 4\,a\,c$? If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent why is $f(t) > 0$ everywhere and why does this imply that one has strict inequality in the usual Schwarz inequality? If $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent with each a negative multiple of the other, explain why we have

$$\langle \mathbf{x},\, \mathbf{y} \rangle \ = \ -\,|\mathbf{x}| \cdot |\mathbf{y}|\ .$$

Finally, explain why there is equality in the weak Schwarz inequality if $\mathbf{x}$ and $\mathbf{y}$ are positive multiples of each other.]

SOLUTION.

We shall follow the hint given above. The function in question can be written out explicitly as

$$f(t) \ = \ t^2\,|\mathbf{y}|^2 \ +\ 2\,t\,\langle \mathbf{x},\, \mathbf{y} \rangle \ +\ |\mathbf{x}|^2$$

and this function is nonnegative for all values of t only if the quadrative polynomial $f(t)$ does not have two distinct real roots. To see this, suppose that one has two distinct real roots $r_1 < r_2$. Since the leading term of t^2 is positive this means that $f(t)$ is positive if $|t|$ is sufficiently large, and in particular that $f''(t)$ is a positive constant. The minimum value of f occurs at $\frac{1}{2}(r_1 + r_2)$, and this is the unique point where $f' = 0$. Thus f' is increasing, so that $f'(r_2) > 0$, which in turn means that f is increasing at r_2, and the latter implies that $f(t)$ must be negative if t lies in the interval

$$[\frac{1}{2}(r_1 + r_2), r_2) \ .$$

Therefore we see that f would have to take negative values for some values if t if f has two distinct real roots.

If a quadratic polynomial with real coefficients does not have two distinct real roots, this implies that the discriminant is less than or equal to zero. Since the discriminant for the polynomial $a\,t^2 + b\,t + c = 0$ is $b^2 - 4\,a\,c$, in our situation this means that

$$\left(2\,\langle \mathbf{x}, \mathbf{y}\rangle\right)^2 \quad \leq \quad 4\,|\mathbf{x}|^2\,|\mathbf{y}|^2 \ .$$

If we take square roots of both sides of this equation we obtain the Schwarz inequality.

The object of this exercise is to determine the conditions under which one has the equation

$$\langle \mathbf{x}\,\mathbf{y}\rangle \quad = \quad |\mathbf{x}| \cdot |\mathbf{y}|$$

and if we square both sides we see that this can only happen if we have equality in the previous inequality displayed above. However, equality in that case corresponds to the vanishing of the discriminant for $f(t)$, which in turn implies that the quadratic equation has a unique root r. By the definition of f, for this choice of r we have $0 = |\mathbf{x} - r\,\mathbf{y}|^2$, which means that $\mathbf{x} = r\,\mathbf{y}$. If r is positive, then one can check directly that one has equality in the weak Schwarz inequality using the string of equations

$$\langle \mathbf{x}, \mathbf{y}\rangle \quad = \quad \langle \mathbf{x}, r\,\mathbf{x}\rangle \quad = \quad r\,\langle \mathbf{x}, r\,\mathbf{x}\rangle \quad = \quad r\,|\mathbf{x}|^2 \quad = \quad |\mathbf{x}| \cdot \left(r\,|\mathbf{x}|\right) \quad = \quad |\mathbf{x}| \cdot |\mathbf{y}| \ .$$

On the other hand, if r is negative, then as suggested in the hint we have

$$\langle \mathbf{x}\,\mathbf{y}\rangle \quad = \quad r\,\langle \mathbf{x}, r\,\mathbf{x}\rangle \quad = \quad -|r|\,\langle \mathbf{x}, \mathbf{x}\rangle \quad < \quad |r|\,|\mathbf{x}|^2 \quad = \quad |\mathbf{x}| \cdot \left(|r|\,|\mathbf{x}|\right) \quad = \quad |\mathbf{x}| \cdot |\mathbf{y}|$$

and hence the only way that one has equality for the weak Schwarz inequality is if the two vectors $\mathbf{x}$ and $\mathbf{y}$ are positive multiples of each other.■

II.1 : Orthogonal bases

Problems from Fraleigh and Beauregard, § $6 - 2$, *pp.* $347 - 349$.

6. Here is the desired orthonormal basis:

$$\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \\ -1/\sqrt{3} \end{bmatrix}, \qquad \begin{bmatrix} 5/\sqrt{51} \\ -1/\sqrt{51} \\ 3/\sqrt{51} \\ 4/\sqrt{51} \end{bmatrix}$$

18. The orthogonal complement is the span of $\mathbf{v}_1 = [1,1,0]$ and $\mathbf{v}_2 = [-3,0,1]$ since by inspection these are two independent vectors orthogonal to $[1,-1,3]$. Applying the Gram-Schmidt process, one obtains the vectors $\mathbf{u}_1 = [1/\sqrt{2}, 1/\sqrt{2}, 0]$ $\mathbf{w} = [-3/2, 3/2, 1]$ and $\mathbf{u}_2 = [-3/\sqrt{23}, 3/\sqrt{23}, 2/\sqrt{23}]$.

Additional exercise

1. Let V be an inner product space and let W be a finite dimensional subspace of V. Prove that $(W^\perp)^\perp = W$.

SOLUTION.

Suppose that $\mathbf{x} \in (W^\perp)^\perp$. Write $\mathbf{x} = \widehat{\mathbf{x}} + \mathbf{x}'$ where the first term lies in W and the second lies in $W^\perp$. By our assumption we know that $\mathbf{x}$ is orthogonal to $\mathbf{x}'$, and therefore we have

$$0 = \langle \mathbf{x}, \mathbf{x}' \rangle \;=\; \langle \widehat{\mathbf{x}}, \mathbf{x}' \rangle + \langle \mathbf{x}', \mathbf{x}' \rangle \;=\; 0 + \langle \mathbf{x}', \mathbf{x}' \rangle$$

which implies that $\mathbf{x}' = \mathbf{0}$; —it i.e., we must have $\mathbf{x} = \widehat{\mathbf{x}} \in W$.

Conversely, if $\mathbf{x} \in W$ then $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for all $\mathbf{y} \in W^\perp$, and therefore we also have $\mathbf{x} \in (W^\perp)^\perp$. Since every vector belongs to one of these subspaces if and only if it belongs to the other, the two subspaces must be equal.■

2. Let $\mathcal{A} = \{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ be an ordered basis for $\mathbf{R}^n$.

(a) Prove that there is a unique *adjoint basis* for $\mathbf{R}^n$ of the form $\mathcal{A}^* = \{\mathbf{a}_1^*, \cdots, \mathbf{a}_n^*\}$ such that $\langle \mathbf{a}_i^*, \mathbf{a}_j \rangle$ is equal to 0 if $i \neq j$ and 1 if $i = j$. [*Hint:* Suppose that $\mathcal{A}$ and $\mathcal{B}$ are ordered bases and that A and B are matrices whose columns display the vectors of $\mathcal{A}$ and $\mathcal{B}$ in the correct order. Then the entries of ${}^{\mathbf{T}}B\,A$ are inner products of vectors in $\mathcal{B}$ with vectors in $\mathcal{A}$.]

(b) If $\mathcal{A}$ and $\mathcal{A}^*$ are as above and $\mathbf{x} \in \mathbf{R}^n$, prove that

$$\mathbf{x} = \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{a}_j^* \rangle \, \mathbf{a}_j \;=\; \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{a}_j \rangle \, \mathbf{a}_j^* \; .$$

SOLUTION.

(a) Follow the hint. The observation about the entries of ${}^{\mathbf{T}}B\,A$ follows because more generally the (i,j) entry of a product matrix PQ is the inner product of the (transpose of the) i^{th} row of P with the the j^{th} column of Q. Therefore a set of n vectors $\mathbf{b}_i$ will be a dual basis if and only if the square matrix B whose columns are given by these vectors (in the proper order) will satisfy ${}^{\mathbf{T}}B\,A = I$, or equivalently if and only if B is the transposed inverse of A. Therefore there is one and only one way that the columns of a square matrix B can represent an adjoint basis; namely, if and only if B is the transposed inverse of A.■

(b) Express $\mathbf{x}$ as a linear combination $\sum_i c_i \, \mathbf{a}_i$ and take inner products with $\mathbf{a}_j^*\rangle$. We then have

$$\langle \mathbf{x}, \mathbf{a}_j^* \rangle \;=\; \sum_i c_i \, \langle \mathbf{a}_i^*, \mathbf{a}_j \rangle$$

and since $\langle \mathbf{a}_i^*, \mathbf{a}_j \rangle$ is equal to 0 if $i \neq j$ and 1 if $i = j$, the expression on right hand side reduces immediately to c_j. This proves the first assertion. To prove the second assertion, notice that if $\mathcal{A}^*$ is an adjoint basis for $\mathcal{A}$, then $\mathcal{A}$ is also an adjoint basis for $\mathcal{A}^*$, and therefore the second expression follows immediately by reversing the roles of $\mathcal{A}$ and $\mathcal{A}^*$ in the preceding argument.■