Math 133
Fall 2018

UPDATED GENERAL INFORMATION — OCTOBER 31, 2018

Here are solutions to the exercises in `aabUpdate06f18.pdf`.

1. Given $\angle abc$ and a line L in the same plane, if these two sets meet in three points then either $L = ab$ or $L = ac$.

SOLUTION. Let x, y, z be the three points. Each of these points lies on either ab or ac, and it follows that at least two of these points lie on one of these lines (this is known as the *pigeonhole principle*). If two or three points lie on ab then $L = ab$ because there is a unique line containing two points. Similarly, if two or three points lie on ac then $L = ac$.■

2. Verify the statement about ruler functions made in the lectures: If L is a line and $h : L \to \mathbb{R}$ is a ruler function, and $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(t) = \varepsilon t + K$ where K is an arbitrary constant and $\varepsilon = \pm 1$, then the composite $g \circ f$ is also a ruler function. — For the sake of completeness, we say that f is a ruler function if it is a 1–1 correspondence from L to $\mathbb{R}$ under which the distances between two points on L equals the distance between the corresponding real numbers (see the first page of Section II.3).

SOLUTION. First of all, f is a misprint and should be replaced by h. — The map $g \circ h$ is a 1–1 correspondence, for an inverse function to g is given by $g^{-1}u = \varepsilon(u - K)$., and therefore the composite of the two 1–1 onto maps h and g has an inverse.

We need to show that $g \circ h$ is distance preserving, and we do this by writing everything out:

$$|gh(x) - gh(y)| \quad = \quad |(\varepsilon h(x) + K) - (\varepsilon h(y) + K)| \quad = \quad |\varepsilon h(x) - \varepsilon h(y)| \quad = \quad |\varepsilon(h(x) - h(y)| \quad =$$

$$|\varepsilon| \cdot |h(x) - h(y)| \quad = \quad |\varepsilon| \cdot \mathbf{d}(x, y)$$

Since $\varepsilon = \pm 1$, its absolute value is one and hence the right hand side is just $\mathbf{d}(x, y)$; therefore $g \circ h$ is distance preserving.■

3. Given ΔABC, let $D \in (AB)$ and $E \in (BC)$, and let $X \in (DE)$. Prove that X and B lie on the same side of AC.

SOLUTION. By construction we have $A * D * B$ and $B * E * C$. These imply that B, D, E all lie on the same side of AC (draw a picture to motivate this if it helps). Now $X \in (DE)$ and by convexity of the two sides of a line, it follows that X lies on the same side of AC as B, D, E.■

4. If P is the plane in $\mathbb{R}^3$ defined by the equation $5x + y + 3z = 7$, determine which of the following points lie on the same side of P as $(1, 0, -1)$ and which do not: $(-1, -4, 9)$, $(-2. -3, 4)$, $(1, -1, 1)$, $(2, -8, 1)$

SOLUTION. Let $f(x, y, z) = 5x + y + 3z$. Then the two sides of the plane are defined by the inequalities $f < 7$ and $f > 7$, and the plane is defined by $f = 7$. So we need to evaluate f at all points in the problem:

$$f(1, 0, -1) = 2 \qquad f(1, 4, 9) = 36 \qquad f(-2, -3, 4) = -1 \qquad f(1, -1, 1) = 7 \qquad f(2, -8, 1) = 5$$

Therefore $(1, 0, -1)$, $(-2, -3, 4)$ and $(2, -8.1)$ are all on one side of the plane, while $(1, 4, 9)$ is on the other and $(1, -1, 1)$ lies on the plane.■

5. Suppose that we are given ΔABC with $B * A * D$ and $B * C * E$, and suppose that $X \in (AC)$. Prove that there is a point Y such that $B * X * Y$.

SOLUTION. We have $A * X * C$, which implies that X lies in the interior of $\angle ABC$, and the latter is equal to $\angle DBE$ because $D \in (BA$ and $E \in BC$. Therefore the Crossbar Theorem implies that the ray $(BX$ meets the open segment (DE) in some point Y. As in Problem 3, betweenness considerations imply that B and X lie on the same side of DE. Since $Y \in (BX$ this means that either $B * X * Y$ or $B * Y * X$. If the second were true then B and X would lie on opposite sides of DE, and hence we must have $B * X * Y$.■

6. Suppose that we are given isosceles triangle ABC in the plane with $|AB| = |AC|$, and let D be such that the ray $[AD$ bisects $\angle BAC$: $|\angle BAD| = |\angle DAC| = \frac{1}{2}|\angle BAC|$. Prove that D is the midpoint of $[BC]$.

SOLUTION. Since $|AD| = |AD|$, the SAS congruence axiom implies that $\Delta BAD \cong \Delta CAD$. This means that $|BD| = |BC|$. By construction D lies in the interior of $\angle CAB$, and therefore we must have $B * D * C$. Therefore D satisfies the conditions for the midpoint of $[BC]$.■