Math 133
Fall 2018

## UPDATED GENERAL INFORMATION — NOVEMBER 30, 2018

*Detailed proofs of some results in neutral geometry*

The two documents described below have been posted; they fill in the details of certain proofs that were not given in Sections V.2 and V.3 of the course notes.

(1) As noted in the lectures, the document `neutral-proofs1.pdf` contains symthetic neutral-geometric proofs of the results listed in Section V.2 of the notes (along with a few additional results). In keeping with the overall pattern of the course, none of the proofs for 3–dimensional neutral geometry will be covered on the third examination.

(2) A complete and detailed proof of Theorem V.3.8 in the course notes has been posted in the file `neutral-rectangles.pdf`.

*Study hints for the second examination*

The new material covered on the examination will come from Sections III.2 – III.5 and V.2 — V.4 of the course notes, with two problems from Unit III and two from Unit V. Two of the problems will involve finding distances or angle measurements whose evaluation requires the use of results from Euclidean geometry covered in the course, and the other two exercises will be traditional synthetic proofs. As is ususally the case in mathematics courses, background information from the first part of the course will be implicit at certain points.

Some things which will not be covered include the last two subsections of Section III.3, the material on excenters in Section III.4 (but the results and definitions involving the four basic concurrence points probably will be!), the subsection *Similarities and linear algebra* in Section III.5, everything in Section V.1, the material up to *How much can one prove without the Fifth Postulate?* in Section V.2, the final two subsections of Section V.3, and the material starting with the subheading *Asymptotic parallels* in Section V.4 (**HOWEVER,** this does not include the problems in the Appendix to V.4!).

*Problems for study*

Here are specific suggestions, including some problems which were considered but either not included or may appear in a simplified form. For the first group, assume that the setting is a Euclidean plane.

(1) If a parallelogram's sides all have equal length, it it a square? Prove this or give a counterexample.

(2) Prove that a parallelogram is a rectangle if and only if the lengths of its diagonals are equal.

(3) Given an isosceles $\Delta ABC$ with $d(A, B) = d(A, C)$ and $\angle BAC$ obtuse, determine whether $d(B, C)$ is longer or shorter than the other two sides and give a proof for your assertion.

(4) Suppose that we are given $A * B * C$ and $D * E * F$. Prove that if $d(A, C) = d(D, F)$ and $d(A, B) = d(D, E)$, then $d(B, C) = d(E, F)$.

(5) Suppose we are given $\Delta ABC$ and $D \in (BC)$ is such that $[AD$ bisects $\angle BAC$. If $|\angle ABC| = x$ and $|\angle ACB| = y$, find $|\angle ADC|$.

(6) Given two lines that are cut by a transversal, give the criteria defining concepts like alternate interior angles, alternate exterior angles and corresponding angles (an exam problem probably will require some understanding beyond simply repeating the definition).

(7) Know the definitions of various polygons, including convex quadrilaterals, parallelograms and trapezoids.

(8) Know the results involving inequalities for the measurements of sides and angles in triangles.

(9) Know the statements of the four concurrence theorems for triangles and the names for the points of concurrency in each one.

Here is a second group of problems covering Sections III.5 —V.4:

(1) Assume that we are working in a given **Euclidean** plane. — Suppose that $\Delta ABC$ is isosceles with $d(A, B) = d(A, C)$, and let $D$ be the midpoint of $[BC]$. Explain why each of the centroid, circumcenter, orthocenter and incenter lie on the line $AD$. Describe examples for which all four of these points lie on $[AD]$, and also describe examples for which at least one of these points does not lie on the open segment $(AD)$.

(2) Assume that we are working in a given **Euclidean** plane. — Suppose that we are given $\Delta ABC$, and let $D \in (AB)$ be such that $\Delta ADC \sim \Delta CDB$. Prove that $\Delta ACB$ is a right triangle, and furthermore we have $\Delta ADC \sim \Delta ACB$ and $\Delta CDB \sim \Delta ACB$.

(3) Assume that we are working in a given **Euclidean** plane. — Suppose that we are given $\Delta ABC$ and $\Delta DEF$, and also assume that there are points $G \in (BC)$, $H \in (EF)$ such that $\Delta ABG \sim \Delta DEH$ and $\Delta AGC \sim \Delta DHF$. Prove that $\Delta ABC \sim \Delta DEF$.

(4) Assume that we are working in a given **Euclidean** plane. — Suppose that we are given lines $L$ and $M$ with distinct points $A, B, C, D, E, F$ such that the first three points are on $L$ and satisfy $A * B * C$, while the second three points are on $M$ and satisfy $D * E * F$. Furthermore, assume that the three lines $AD$, $BE$ and $CF$ are all parallel to each other. Prove that

$$\frac{d(A,B)}{d(B,C)} \quad = \quad \frac{d(D,E)}{d(E,F)} \ .$$

[*Hint:* Use vectors and write $C = A + t(B - A)$ and $F = D + u(E - D)$ for suitable scalars. What do the betweenness relations say about $t$ and $u$ separately? How do the parallelism conditions imply an equation involving $t$ and $u$?]

Also consider the following converse problem: If two of the three lines are parallel and the proportionality equation is valid, prove that the third line is parallel to the other two.

(5) Assume that we are working in a given **neutral** plane. — Suppose that $\Delta ABC$ is isosceles with $d(A, B) = d(A, C)$, let $E$ and $F$ be the midpoints of $[AB]$ and $[AC]$ respectively, and suppose that $D$ is the midpoint of $[BC]$. Prove that the lines $AD$ and $EF$ are perpendicular. [*Note:* At some point in the proof it will probably be necessary to show that $AD$ and $EF$ have a point in common.]

(6) Assume that we are working in a given **neutral** plane. — Suppose that $\Delta ABC$ is given and that $B * C * D$ is true. Prove that $|\angle BCD| \geq |\angle BAC| + |\angle ABC|$. Can we state a stronger conclusion if the plane is hyperbolic? Give reasons for your answer.

(7) Assume that we are working in a given **hyperbolic** plane. — Using the exercises for Section V.3 and results from Section V.4, explain why a Saccheri quadrilateral is never a Lambert quadrilateral and vice versa. [*Hint:* In neutral geometry, explain how the exercises imply that a convex quadrilateral which is both a Saccheri quadrilateral and a Lambert quadrilateral must be a rectangle.]

(8) Suppose we are given a circle $\Gamma$ in a **neutral** plane, and suppose that $A, B, C \in \Gamma$ are such that neither the line $AB$ nor the line $BC$ contains the center $Q$ of the circle. Prove that $Q$ lies on the perpendicular bisectors of $[AB]$ and $[BC]$.

(9) Assume that we are working in some **Euclidean** plane. — Suppose that we are given real numbers $0 < a < b$. Explain why there is a triangle in the plane whose sides have length $a$, $2a$ and $b$ if and only if $b < 3a$.

Finally, here is a third group of problems:

(1) Assume that we are working in a given **neutral** plane. — Suppose that $L$ and $M$ are parallel lines, and suppose that $X$ is a point such that $(i)$ the point $X$ and line $M$ are on the same side of $L$, $(ii)$ the point $X$ and the line $L$ are on the same side of $M$, and $(iii)$ $X \in AB$ where $A \in L$ and $B \in M$. Prove that we must have $A * X * B$.

(2) Assume that we are working in a given **neutral** plane. — Suppose that we are given two side-by-side Saccheri quadrilaterals $ABCD$ and $DCEF$ such that $B * C * E$, each of $AB$, $CD$, $EF$ is perpendicular to the line of $B, C, E$, the points $A, D, F$ all lie on the same side of this line, and $|AB| = |CD| = |EF|$. Prove that $A, D, F$ are collinear if the plane is Euclidean, and $A, D, F$ are noncollinear if the plane is hyperbolic.

For the remaining problems, assume that we are working in a given Euclidean plane.

(3) Let $\Delta ABC$ be a right triangle with a right angle at $B$ such that $|BC| = 12$ and $|AC| = 15$. Now let $D$ and $E$ be points such that $A * B * D$, $A * C * E$, $DE \perp AB$, and $|DE| = 30$. Find $|CE|$.

(4) Suppose that $A, B, C, D$ (in that order) are the vertices of a parallelogram, and let $E \in (BC)$ such that $(AC)$ meets $(DE)$ at a (unique) point $F$, $|AD| = 18$, $|AF| = 24$, and $|BE| = 6$. Find $|FC|$. [*Hint:* Look at the alternate interior angle pairs associated to the transversals $AC$ and $DE$.]

Illustrations for the last group of problems will be posted in a companion file to this one. Solutions will be posted very early next week.