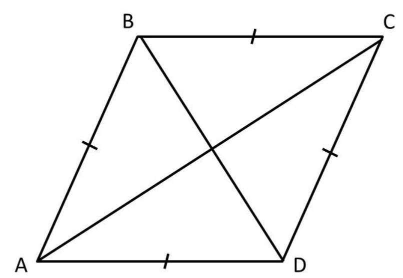
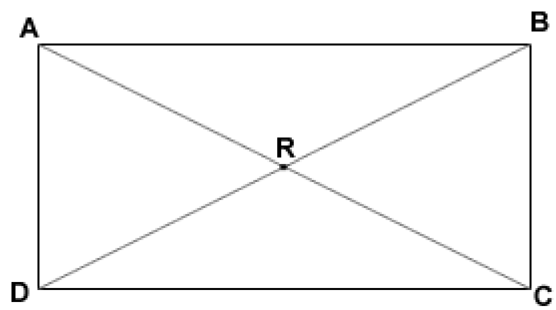
Solutions for the first problem group

1. Not necessarily. See the picture below.



More precisely, if the four sides have a fixed length **L** and α, β are positive numbers adding up to $180°$, then there is a parallelogram whose sides all have length **L** and whose opposite pairs of angles have measures α and β.

2. Suppose first that we have a rectangle as below, with right angles at all four vertices. Then |**AB**| = |**CD**| and |**AD**| = |**BC**| imply the right triangle congruence $\triangle$**ADC** $\cong$ $\triangle$**BCD** and hence |**AC**| = |**BD**|.



Conversely, suppose that |**AC**| = |**BD**|. Since the diagonals of a parallelogram bisect each other we have |**AR**| = |**RC**| = ½ |**AC**| = ½ |**BD**| = |**BR**| = |**RD**|. By the Vertical Angle Theorem we have |$\angle$**ARB**| = |$\angle$**DRC**| and |$\angle$**ARD**| = |$\angle$**BRC**|, and therefore by **SAS** we also have $\triangle$**ARB** $\cong$ $\triangle$**DRC** and $\triangle$**ARD** $\cong$ $\triangle$**BRC.** The second sentence in this paragraph implies that each of these triangles is isosceles such that the two sides of equal length meet at **R,** and therefore we also have the following:

$$|\angle \mathbf{RAB}| = |\angle \mathbf{RBA}| = |\angle \mathbf{RCD}| = |\angle \mathbf{RDC}| \quad \text{and}$$

$$|\angle \mathbf{RAD}| = |\angle \mathbf{RDA}| = |\angle \mathbf{RBC}| = |\angle \mathbf{RCB}|.$$

The additivity property for angle measure implies that the sum of these eight quantities is equal to the angle sum for the vertex angles of the parallelogram **ABCD** and hence equals $360°$. In fact, we have the following four sequences of equations:

$$|\angle \mathbf{ADC}| = |\angle \mathbf{ADR}| + |\angle \mathbf{RDC}| = |\angle \mathbf{ADR}| + |\angle \mathbf{RDC}|$$

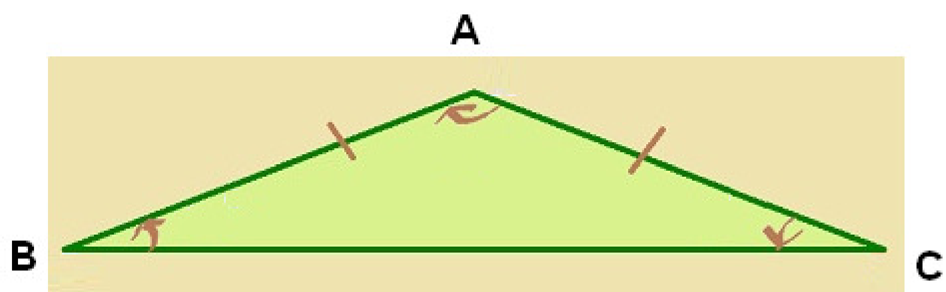
$$|\angle \mathbf{DCB}| = |\angle \mathbf{DCR}| + |\angle \mathbf{RCB}| = |\angle \mathbf{ADR}| + |\angle \mathbf{RDC}|$$

$$|\angle \mathbf{CBA}| = |\angle \mathbf{CBR}| + |\angle \mathbf{RBA}| = |\angle \mathbf{ADR}| + |\angle \mathbf{RDC}|$$

$$|\angle \mathbf{BAD}| = |\angle \mathbf{BAR}| + |\angle \mathbf{RAD}| = |\angle \mathbf{ADR}| + |\angle \mathbf{RDC}|$$

It follows that all four vertex angles of the parallelogram have equal measures. Since two consecutive angles of a parallelogram are supplementary, this means that all four vertex angles must be $90°$ angles and therefore the parallelogram is a rectangle.

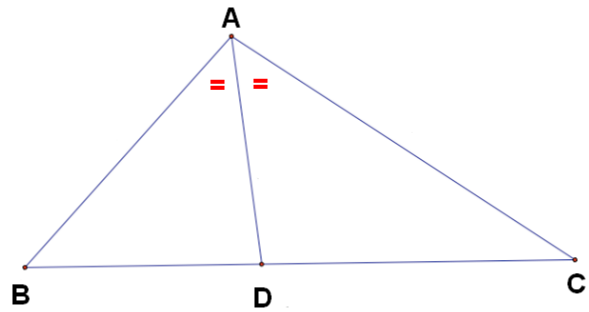
3. The picture suggests that **[BC]** is the longest side. This will follow if we can show that the measure of $\angle$**BAC** is greater than the measures of $\angle$**ABC** and $\angle$**ACB** (which are equal by the Isosceles Triangle Theorem).



Now the angle sum of a triangle is equal to $180°$, and hence at least two angles must be acute. We have assumed that the vertex angle at **A** is obtuse, so therefore the angles at **B** and **C** must be acute. The desired conclusion now follows because the larger angle is opposite the longer side.

4. The betweenness assumptions **A**$*$**B**$*$**C** and **D**$*$**E**$*$**F** imply that $|\mathbf{AC}| = |\mathbf{AB}| + |\mathbf{BC}|$ and $|\mathbf{DF}| = |\mathbf{DE}| + |\mathbf{EF}|$. Equivalently, we have that $|\mathbf{BC}| = |\mathbf{AC}| - |\mathbf{AB}|$ and $|\mathbf{EF}| = |\mathbf{DF}| - |\mathbf{DE}|$. Since we are given that $|\mathbf{AC}| = |\mathbf{DF}|$ and $|\mathbf{AB}| = |\mathbf{DE}|$, the desired conclusion follows by subtracting the second of these equations from the first.

5. The drawing below may be helpful for understanding the solution**.**



We are given that $|\angle ABC| = x$ and $|\angle ACB| = y$, so that $|\angle BAC| = 180 - x - y$. Since **[AD** is a bisector, we have $|\angle DAC| = 90 - ½x - ½y$. Therefore we have that

$$|\angle ADC| = 180° - |\angle ACD| - |\angle DAC|$$

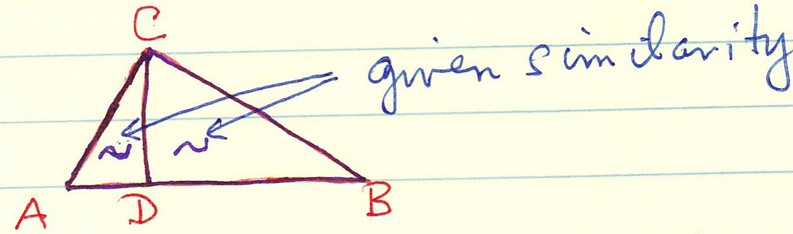
and if we substitute the values in the preceding two sentences we find that $|\angle ADC|$ is equal to $90 + ½x - ½y$.

Hints for the problems in the second group

[redacted]

1. AD is the perpendicular bisector of [BC], and [AD bisects ∡BAC. — To find examples where some points do not lie on (AD), take |∡BAC| close to 180°. (see page 3)

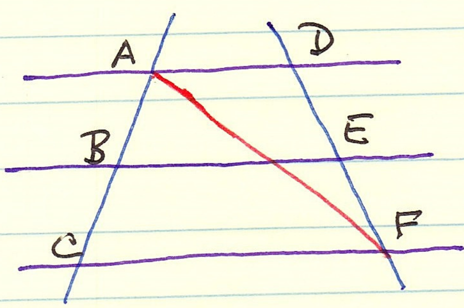
2.



First show AB ⊥ CD, then show that AC ⊥ BC.

3. Show that the ratios of similitude for the two similarities are the same (note that [AG] and [DH] are sides of <u>two</u> triangles in the hypotheses). Using the equality of the ratios, imitate the proof of Problem 1 on Examination 2.

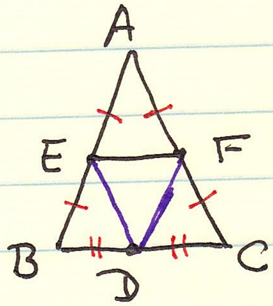
4.



This drawing may be helpful.

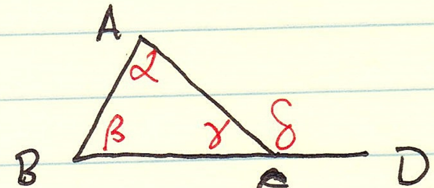
What does the parallelism condition in the converse problem imply about E−B and D−A?

<u>vector subtraction</u>

5. 

Why are $d(A,E)$, $d(E,B)$, $d(A,F)$ and $d(F,C)$ all equal?

Prove that $\Delta EBD \cong \Delta FCD$. Why does this imply that AD is the perpendicular bisector of $[EF]$?

6. 

$\gamma + \delta = 180^\circ$

What does the Saccheri-Legendre Theorem imply? What are the stronger conclusions that hold if the plane is Euclidean or hyperbolic?

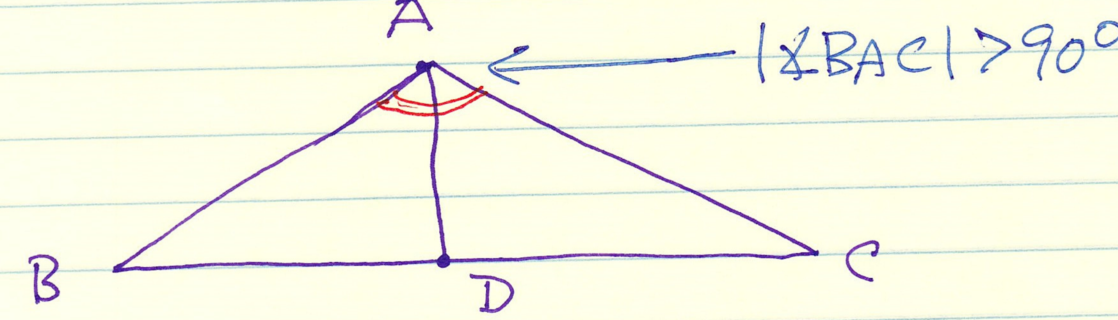
7. How many acute angles are there in a $\left\{\begin{matrix}\text{Lambert}\\ \text{Saccheri}\end{matrix}\right\}$ quadrilateral?

(also see p.3)

8. The point Q is equidistant from A, B, C.

9. For one direction, this follows from a result in Section III. 2; for the other, this follows from a result in Section III. 6.

Additional drawings for #1



Neither the circumcenter nor the orthocenter lies on (AD) — where do they lie?

Note that both the centroid and incenter ALWAYS lie on (AD).

FOOTNOTE FOR #8

To see that the perpendicular bisectors of AB & BC are distinct, notice that if they were the same line L, then $L \perp AB$ & $L \perp BC$ would imply $AB \parallel BC$ or $AB = BC$, and neither of the latter is true.