Mathematics 133, Fall 2013, Examination 1

Answer Key

1. [*25 points*] Find the barycentric coordinates of the point $P = (4, 7)$ with respect to $A = (1, 1)$, $B = (0, 1)$ and $C = (1, 0)$.

SOLUTION

The main step is to write $P - A = y(B - A) + z(C - A)$ for suitable real numbers y and z; if $x = 1 - y - z$, then $P = xA + yB + zC$ is the barycentric coordinates expansion of P with respect to A, B, C.

We have $P - A = (3, 6)$, $B - A = (-1, 0)$ and $C - A = (0, -1)$, from which we may read off the expansion $P - A = (-3) \cdot (B - A) + (-6) \cdot (C - A)$. Therefore $y = -3$, $z = -6$, and $x = 1 - (-3) - (-6) = 11$.■

2. [*25 points*] (a) State the defining condition for a convex subset in the coordinate plane $\mathbf{R}^2$.

(b) Given three noncollinear points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in $\mathbf{R}^2$ and an arbitrary vector $\mathbf{v} \in \mathbf{R}^2$, what condition on the barycentric coordinates of $\mathbf{v}$ (with respect to $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$) is equivalent to the statement that $\mathbf{v}$ and $\mathbf{c}$ lie on the same side of the line $\mathbf{ab}$?

(c) Given a line L in the coordinate plane and two points $\mathbf{a}$ and $\mathbf{b}$ on opposite sides of L, what conclusion can we draw concerning the lines $\mathbf{ab}$ and L?

SOLUTION

(a) A subset K is convex if for all $\mathbf{x}$ and $\mathbf{y}$ in K and $0 \leq t \leq 1$ we have $t\,\mathbf{x}+(1-t)\mathbf{y} \in K$, or equivalently the closed segment $[\mathbf{xy}]$ is contained in K (or equivalently the first condition with strict inequalities or the second with the closed segment replaced by an open segment; any one of these alternatives is a correct answer).■

(b) The barycentric coordinate of $\mathbf{v}$ with respect to $\mathbf{c}$ is positive if and only if the two points lie on the same side of the line.■

(c) The two lines must have a point in common. In fact, the common point lies on the open segment $(\mathbf{ab})$.■

3. [*25 points*] Given two points A and B in the coordinate plane, let $C = \frac{1}{2}A + \frac{1}{2}B$ denote their midpoint, which lies on the open segment (AB). Let $X = A + t(B - A)$ be an arbitrary point of the line AB. For which values of t does X lie on the closed ray $[CB$?

SOLUTION

A rough sketch suggests that the point lies on $[CB$ if and only if $t \geq \frac{1}{2}$. Here is one way of verifying this algebraically by systematically applying the main result in the document `betweenness.pdf`:

If $X \in [BC$ then $t \geq \frac{1}{2}$:

> If $X = C$ then $t = \frac{1}{2}$ while if $X = B$ then $t = 1$.
>
> If $C * X * B$ then $\frac{1}{2} < t < 1$.
>
> If $B * C * X$ then $\frac{1}{2} < 1 < t$ (note that the first inequality can be omitted without affecting the validity of the reasoning).

If $X \notin [BC$ then $t < \frac{1}{2}$:

> In this case we have $X * B * C$, which implies that $t < \frac{1}{2} < 1$ (note that the second inequality can be omitted without affecting the validity of the reasoning).■

4. [*25 points*] Let $A = (0, 3)$, let $B = (2, 4)$, and let L be the line AB. Determine which two of the points $(10, 9)$, $(9, 7)$ and $(8, 6)$ lie on the same side of L (any valid method is acceptable; in particular, you do not have to use the criterion in part (b) of Problem 2).

SOLUTION

Probably the easiest way to solve this problem is to start by finding the equation of the line AB in the form $y = mx + b = g(x)$, so that the two sides of the line are defined by the inequalities $y > g(x)$ and $y < g(x)$. We can then compare y and $g(x)$ for each of the three given points to decide which lie on each side of AB.

Direct computation shows that the **slope** of the line is

$$m = \frac{4-3}{2-0} = \frac{1}{2}$$

and the y–**intercept** b is 3, so that $g(x) = \frac{1}{2}x + 3$ is the linear function whose graph is AB.

For each point (x, y) we now compute $g(x)$ and compare it to y:

For $(10, 9)$ we have $g(10) = 8 < 9 = y$.

For $(9, 7)$ we have $g(9) = 7\frac{1}{2} > 7 = y$.

For $(8, 6)$ we have $g(8) = 7 > 6 = y$.

These inequalities show that $(9, 7)$ and $(8, 6)$ lie on one side of AB and $(10, 9)$ lies on the other side of AB.■