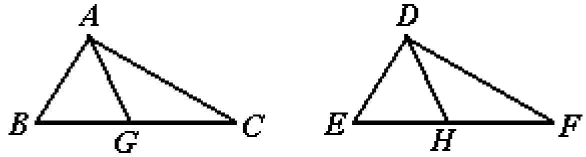
1. **[*20 points*]** Suppose that we have $\triangle ABC$ and $\triangle DEF$ in the Euclidean plane and points G and H on (BC) and (EF) respectively such that $\triangle ABG \cong \triangle DEH$ and $\triangle AGC \cong \triangle DHF$. Prove that $\triangle ABC \cong \triangle DEF$.



SOLUTION

The first congruence assumption implies that $d(A, B) = d(D, E)$, $d(B, G) = d(E, H)$, and $d(A, G) = d(D, H)$. The second congruence assumption implies that we also have $d(A, C) = d(D, F)$ and $d(G, C) = d(H, F)$.

The conditions on G and H imply that $B*G*C$ and $E*H*F$. If we combine this with the preceding two sentences we obtain the equation $d(B, C) = d(B, G) + d(G, C) = d(E, H) + d(H, F) = d(E, F)$.

Taken together, the preceding equations imply that $\triangle ABC \cong \triangle DEF$ by the **SSS** criterion for triangle congruence.■

2. **[*25 points*]** Suppose that we have an isosceles triangle $\triangle ABC$ in the Euclidean plane such that the two sides meeting at A have the same length and $\alpha = |\angle BAC| < 60°$. Which edge(s) of $\triangle ABC$ is/are the shortest one(s)? Give reasons for your answer. **[*Hint:*** How do the measures of the other vertex angles compare to α?**]**

SOLUTION

We are given that $d(A, B) = d(A, C)$, and therefore $|\angle ABC| = |\angle ACB|$ by the Isosceles Triangle Theorem.

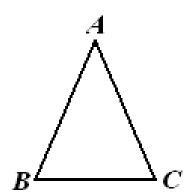
Since the angle sum of a triangle is $180°$, the angle measurement equation in the first sentence implies that $180° = \alpha + |\angle ABC| + |\angle ACB| = \alpha + 2|\angle ABC| = \alpha + 2|\angle ACB|$.

Dividing the preceding equations by 2 and rearranging terms, we find that $|\angle ABC| = |\angle ACB| = 90° - \frac{1}{2}\alpha$.

Since $\alpha < 60°$, the preceding sentence implies that $|\angle ABC| = |\angle ACB| > 60° > \alpha$.

Finally, since the longest side of a triangle is opposite the largest angle, we have $d(A, B) = d(A, C) > d(B, C)$.■

Here is a drawing of a typical example:



3. [***15 points***] Suppose that we have two lines ***BD*** and ***CE*** in the Euclidean plane which are cut by the transversal ***BC*****,** and let ***A*** be a point such that ***A∗B∗C*****.**

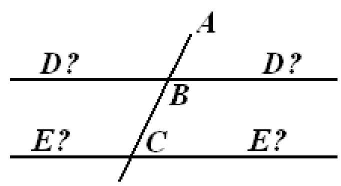
(***a***) If ∠***DBC*** and ∠***BCE*** are **alternate interior angles,** do ***D*** and ***E*** lie on the same or on opposite sides of ***BC*****?**

(***b***) If ∠***ABD*** and ∠***BCE*** are **corresponding angles,** do ***D*** and ***E*** lie on the same or on opposite sides of ***BC*****?**

SOLUTION

(***a***) By the definition of alternate interior angles**,** the points ***D*** and ***E*** lie on **opposite sides** of ***BC*****.**■

(***b***) By the definition of corresponding angles**,** the points ***D*** and ***E*** lie on **the same side** of ***BC*****.**■



4. **[*15 points*]** Suppose that are given a parallelogram ***ABCD*** in the Euclidean plane**.**

(***a***) Show that ***C*** and ***D*** are on the same side of the line ***AB.***

(***b***) Explain why ***A*, *B*, *C*** and ***D*** (in that order) form the vertices of a convex quadrilateral**.** **[*Hint:*** Three other statements similar to the first one are true**.** What are they**?]**

SOLUTION

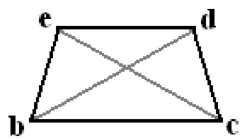
(***a***) Since ***AB*** and ***CD*** are parallel**,** neither ***C*** nor ***D*** lies on ***AB*.** Furthermore**,** if ***C*** and ***D*** were on opposite sides of ***AB*,** then by the Plane Separation Postulate the lines ***CD*** and ***AB*** would have a point in common**,** and we are given that they do not**.** Therefore ***C*** and ***D*** must lie on the same side of ***AB*.**■

(***b***) Consider what happens if we permute the roles of the vertices as follows**:** If we make the substitution $A_1 = B$, $B_1 = C$, $C_1 = D$ and $D_1 = A$, then the conclusion of (***a***) translates into a statement that ***D*** and ***A*** are on the same side of the line ***BC.*** Similarly, if we make the substitution $A_1 = C$, $B_1 = D$, $C_1 = A$ and $D_1 = B$, then the conclusion of (***a***) translates into a statement that ***A*** and ***B*** are on the same side of the line ***CD.*** Finally, if we make the substitution $A_1 = D$, $B_1 = A$, $C_1 = B$ and $D_1 = C$, then the conclusion of (***a***) translates into a statement that ***B*** and ***C*** are on the same side of the line ***AD.*** Combining these with **(*a*),** we see that ***A*, *B*, *C*** and ***D*** (in that order) form the vertices of a convex quadrilateral.■

Alternate solution to parts (*a*) and (*b*): By vector geometry we know that the open diagonals **(*AC*)** and **(*BD*)** of the given parallelogram meet at a common midpoint ***X*.** This implies that ***A*∗*X*∗*C*** and ***B*∗*X*∗*D*** hold**,** which in turn implies that

A*, *X*,** and ***B lie on the same side of ***CD*,**
B*, *X*,** and ***C lie on the same side of ***AD***,
C*, *X*,** and ***D lie on the same side of ***AB*,** and last but not least
D*, *X*,** and ***A lie on the same side of **BC.**■

5. **[*25 points*]** Suppose that we have an isosceles trapezoid $\square\,\mathbf{bcde}$ in the coordinate plane such that the vectors $\mathbf{b}$ and $\mathbf{c}$ are linearly independent, $|\mathbf{b}| = |\mathbf{c}|$, and s is a scalar such that $0 < s < 1$ with $\mathbf{d} = s\,\mathbf{c}$ and $\mathbf{e} = s\,\mathbf{b}$. The diagonals $[\mathbf{bd}]$ and $[\mathbf{ce}]$ of the trapezoid are known to meet at some point $\mathbf{p}$ which satisfies $\mathbf{p} = t\,\mathbf{b} + (1-t)\,\mathbf{d} = t\,\mathbf{c} + (1-t)\,\mathbf{e}$ for some scalar t. Express t in terms of s.



SOLUTION

If we make the substitutions $\mathbf{d} = s\,\mathbf{c}$ and $\mathbf{e} = s\,\mathbf{b}$ in the equation $\mathbf{p} = t\,\mathbf{b} + (1-t)\,\mathbf{d} = t\,\mathbf{c} + (1-t)\,\mathbf{e}$ we obtain the new equation $t\,\mathbf{b} + (1-t)s\,\mathbf{c} = t\,\mathbf{c} + (1-t)s\,\mathbf{b}$, and since $\mathbf{b}$ and $\mathbf{c}$ are linearly independent the coefficients of $\mathbf{b}$ and $\mathbf{c}$ on both sides of the equation must be equal.

Therefore we have $(1-t)s = t$. If we solve this for t in terms of s we obtain $t = s/1+s$.■