AFFINE TRANSFORMATIONS, AREAS AND VOLUMES

In this note, if E is a "reasonable" region in $\mathbb{R}^2$ or $\mathbb{R}^3$ for which an area or volume can be defined, then the area or volume of E will be called the **measure** of E and denoted by $\mu(E)$.

The starting point is the following natural question:

> *Suppose that E is a "reasonable" subset of $\mathbb{R}^n$, where $n = 2$ or 3, and suppose that $T(\mathbf{x}) = A\,\mathbf{x} + \mathbf{b}$, where A is an invertible linear $n \times n$ and $\mathbf{b}$ is some vector in $\mathbb{R}^n$. What is the relationship between $\mu(E)$ and $\mu(T[E])$?*

Direct calculation shows that the Jacobian determinant for T is equal to the determinant $\det A$, and therefore the standard Change of Variables Formula from multivariable calculus implies that the measures of E and $T[E]$ are related by the formula

$$\mu(T[E]) \quad = \quad |\det A| \cdot \mu(E) \ .$$

In view of this, we shall say that an affine transformation T as above is *measure–preserving* if $|\det A| = 1$, and this condition is equivalent to the identity $\mu(T[E]) = \mu(E)$ for the "reasonable" subsets of $\mathbb{R}^n$ (the determinant condition implies the equality of the measures, and conversely if $|\det A| \neq 1$ then the Change of Variables Formula implies that $\mu(T[E]) \neq \mu(E)$ when E is the standard square or cube defined by the coordinate inequalities $0 \leq x_i \leq 1$ for $i = 1\ ,\cdots,\ n$).

Our goal is to prove the following result:

THEOREM. *If T_1 and T_2 are measure-preserving affine transformations of $\mathbb{R}^n$, then the composite $T_1 \circ T_2$ is also a measure-preserving affine transformations. Furthermore, if T is a measure-preserving affine transformations of $\mathbb{R}^n$, then its inverse T^{-1} is also a measure-preserving affine transformation.*

Proof. We shall first prove the result on composites. If we write the affine transformations in the form

$$T_i(\mathbf{x}) \quad = \quad A_i\,\mathbf{x} \ + \ \mathbf{b}_i \ , \qquad (i = 1, 2)$$

then we may write the composite as

$$T_1 \circ T_2(\mathbf{x}) \quad = \quad A_1 A_2\,\mathbf{x} \ + \ (A_1\,\mathbf{b}_2 + \mathbf{b}_1)$$

which shows that the composite is also an affine transformation and its Jacobian is equal to $\det A_1 A_2 = \det A_1 \cdot \det A_2$. If T_1 and T_2 are measure-preserving, then $|\det A_1| = |\det A_2| = 1$ implies $|\det A_1 A_2| = |\det A_1| \cdot |\det A_2| = 1 \cdot 1 = 1$, so it follows that the composite is also measure-preserving.

Next, we shall prove the result for inverses. If we write the affine transformations in the form

$$T(\mathbf{x}) \quad = \quad A\,\mathbf{x} \ + \ \mathbf{b}$$

then we may write the inverse as

$$T^{-1}(\mathbf{y}) \quad = \quad A^{-1}\,\mathbf{y} \ - \ A^{-1}\,\mathbf{b}$$

which shows that the inverse is an affine transformation and its and the Jacobian of the inverse is $\det A^{-1} = (\det A)^{-1}$. If T is measure-preserving, then $|\det A^{-1}| = |\det A|^{-1}$ implies that $|\det A^{-1}| = |\det A|^{-1} = 1^{-1} = 1$ so it follows that the inverse is also measure-preserving.■

Similar considerations apply to "reasonable" regions in $\mathbb{R}^n$ (for all $n \geq 1$) such that a suitable notion of *hypervolume* can be defined (for example, the hypervolume of the **hypercube** $0 \leq x_i \leq a$ in $\mathbb{R}^n$ is a^n).