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Mathematics 133, Fall 2018, Examination 1

Answer Key

1. [*25 points*] Let a and b be real numbers with $a = \pm 1$. Verify that the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax + b$ is distance preserving: $|f(x) - f(y)| = |x - y|$ for all real numbers x and y.

SOLUTION

Write out the left hand side and show that it reduces to the right hand side:

$$|f(x) - f(y)| \quad = \quad |(ax + b) - (ay + b)| \quad = \quad |ax - ay| \quad = \quad |a| \cdot |x - y|$$

Since $a = \pm 1$ we have $|a| = 1$ and hence the right hand side is equal to $|x - y|$, and this proves the assertion in the problem.∎

2. [*25 points*] Let x, y, z, w be noncoplanar points in 3-space (hence no three are collinear). Show that the lines xy and zw have no points in common but are not coplanar.

SOLUTION

The lines are not coplanar, for if they were then a plane containing both of them would contain x, y, z, w. If the lines xy and zw had some point(s) in common then they would be contained in some plane, and this cannot happen by the preceding sentence.■

3. [*25 points*] Let a, b, c, d, e be points in the plane such that $ab \neq ad$, $a * b * c$ and $a * d * e$. Using Pasch's Theorem for triangle acd, show that the open segments (cd) and (be) have a point in common. [*Hint:* Consider the line eb. Why does it not contain any points of $[ad]$? A rough sketch should provide some insight.]

SOLUTION

We shall use the suggestions in the hint, and we shall verify that the four lines are distinct after completing the solution,

Since be meets open edge (ac) in a point, Pasch's Theorem implies that the line eb contains a point from one of the sets (ad), $\{d\}$ or (cd). It cannot contain a point from either of the first two sets, for the lines be and $ad = ae$ are distinct and meet at e, which is not in either (ad) or $\{d\}$. Therefore be and (cd) must have a point x in common.

If we switch the roles of (b, d) and (c, e) in the preceding argument, we also see that cd and (be) must have a point y in common. Now $cd \neq be$, for if they were the same line L then b, c, d, e would all lie on L, and this would mean that $a \in L$ as well. But $cd \neq be$ and hence these lines have at most one point in common. Therefore $x = y$. By the preceding discussion we know that this point lies on both (cd) and (be).$\blacksquare$

PROOF THAT THE FOUR LINES ARE DISTINCT

By construction we know that $ab \neq ad$. Also, be is equal to neither of these lines because it meets the first one at $b \neq a$ and the second one at $e \neq a$. Similarly we know that cd is equal to neither ab or ad. The only remaining possibility is that $cd = be$. This can also be eliminated because cd meets ab in c while be meets ab in b, and the points b and c are distinct by construction; if $cd = be$, then there would be a single point where this line met ab.$\blacksquare$

4. [*25 points*] Suppose we are given isosceles triangle ABC in the plane with $|AB| = |AC|$, and let D be the midpoint of $[BC]$. Prove that the ray $[AD$ bisects $\angle BAC$: $|\angle BAD| = |\angle DAC| = \frac{1}{2}|\angle BAC|$. You may assume D lies in the interior of $\angle BAC$ without proving this fact.

SOLUTION

By definition of the midpoint, we have $|BD| = |DC|$. Therefore by the SSS congruence axiom we know that $\Delta ADC \cong \Delta BDC$, so that $|\angle BAD| = |\angle CAD|$. Call this value q.

Since D is the midpoint of $[BC]$ we have $B * D * C$, so that D lies in the interior of $\angle BAC$. Therefore the additivity axiom implies that

$$|\angle BAC| \quad = \quad |\angle BAD| \; + \; |\angle CAD| \quad = \quad q + q \quad = \quad 2q \; .$$

Combining these observations we see that $|\angle BAD| = |\angle CAD| = q = \frac{1}{2}\,|\angle BAC|$.■