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Mathematics 133, Fall 2018, Examination 2

Answer Key

1. [*25 points*] Suppose that ΔABC is a right triangle with a right angle at B such that $|AB| = 15$, $|BC| = 20$ and $|AC| = 25$. Choose $D \in (AC)$ so that $|AD| = 10$ and $|BC| = 15$, and let $E \in (BC)$ be such that $DE \perp AC$ (one can show that such a point exists, but it is not necessary to do so here). Find the numerical value of $x = |DE|$. — **NOTE:** There is a drawing for this problem on the last page.

SOLUTION

We know that $\angle DCE = \angle BCA$ ∗**as sets**∗ (hence their angle measures are equal) and $|\angle EDC| = 90° = |\angle ABC|$. Therefore $\Delta EDC \sim \Delta ABC$ by **AA** similarity. This yields the proportionality relation

$$\frac{|ED|}{|AB|} = \frac{|DC|}{|BC|} .$$

If we substitute the values for the lengths of the various segments we find that

$$\frac{x}{15} = \frac{15}{20} = \frac{3}{4}$$

so that $4x = 45$ or $x = 45/4$.■

2. [*25 points*] Let ΔABC be an isosceles right triangle with vertices $A = (-1, 0)$, $B = (0, 1)$ and $C = (1, 0)$. Find the circumcenter, orthocenter and centroid for ΔABC, and describe the line which contains all three of these points.

Extra credit 5 points. Explain why this line also contains the incenter of the triangle.

SOLUTION

Since we have an isosceles right triangle the ortheocenter is at the right angle vertex, which is $(0, 1)$, and the circumcenter is the midpoint of the hypotenuse, which is $(0, 0)$. Finally, the centroid is just $\frac{1}{3} \cdot (\,(-1, 0) + (0, 1) + (1, 0)\,) \;=\; (0, 1/3)$.■

For the second part, the incenter lies on the angle bisector, which in this case is just the y-axis, and we have seen that this line contains the other three concurrency points.■

3. [*20 points*] Assume we are working in neutral plane geometry, and let A, B, C, D be the vertices of a Saccheri quadrilateral with $AB \perp BC \perp CD$ and $|AB| = |CD|$. If the lengths of the remaining two sides satisfy $|BC| = |AD|$, prove that this quadrilateral is a rectangle.

SOLUTION

The hypothesis on lengths implies that $\Delta BAD \cong \Delta DCB$ and $\Delta ADC \cong \Delta CBA$ by **SSS** congruence. These congruences yield $90^\circ = |\angle DCB| = |\angle BAD|$ and $90^\circ = |\angle CBA| = |\angle ADC|$. Therefore all four vertex angles for the quadrilateral are right angles and hence the quadrilateral is a rectangle.■

4. [*30 points*] Assume we are working in hyperbolic plane geometry and we are given an equilateral triangle ΔABC (so $|AB| = |AC| = |BC|$). Let D and E be points such that $A * B * D$, $A * C * E$, and $|AD| = 2\,|AB| = 2\,|AC| = |AE|$ (hence ΔADE is isosceles). Prove that $|\angle ADE| < |\angle ABC|$ and $|\angle AED| < |\angle ACB|$. Why does this imply that ΔADE is not equilateral? [*Hint:* Compute angle defects for the relevant triangles, and recall that equilateral $\Leftrightarrow$ equiangular).]

SOLUTION

Since the first triangle is isosceles we know that the measures for all three of its vertex angles are equal to some value, say α. We also know that ΔADE is isosceles, so that $|\angle ADE| = |\angle AED| = \beta$ for some β. Therefore the angle defects of the two triangles satisfy

$$180^\circ - 3\alpha = \delta(\Delta ABC)\,, \quad 180^\circ - \alpha - 2\beta = \delta(\Delta ADE)\,.$$

But we also have $\delta(\Delta ABC) < \delta(\Delta ADC) < \delta(\Delta ADE)$ by the additivity of the angle defect. Combining the conclusions in the preceding two sentences we find that $180^\circ - 3\alpha < 180^\circ - \alpha - 2\beta$, which reduces to $\beta < \alpha$.

Since $\angle DAE = \angle BAC$ ∗**as sets**∗, the measures of the vertex angles for ΔADE are α, β, β. Since equilateral $\Leftrightarrow$ equiangular holds for triangles in neutral geometry, it follows that ΔADE cannot be equilateral.■