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Mathematics 133, Winter 2009, Examination 2

Answer Key

1. [*15 points*] Suppose that the points A, B and C have coordinates $(1,1)$, $(0,0)$ and $(2,1)$ respectively. Using barycentric coordinates, show that $(4,3)$ lies in the interior of $\angle ABC$.

SOLUTION.

To find the barycentric coordinates of $(4,3)$ we first need to solve the equation

$$(4,3) \; - \; B \quad = \quad u(A-B) \; + \; w(C-B)$$

for u and w. Since $B = \mathbf{0}$, this equation is just

$$(4,3) \quad = \quad u(1,1) \; + \; w(2,1)$$

and if we solve for u and w we obtain $w = 1$ and $u = 2$. The third barycentric coordinate is given by $v = 1-u-w = 1-3 = -2$, but we really do not need this because the conditions $u, w > 0$ already imply that $(4,3)$ lies in the interior of $\angle ABC$.

2. [*20 points*] Suppose that we are given $\angle ABC$ and $\angle DEF$ such that $|\angle ABC| = |\angle DEF|$, and let $G \in \text{Int}\angle ABC$ and $H \in \text{Int}\angle DEF$ be such that $|\angle ABG| = |\angle DEH|$. Prove that $|\angle CBG| = |\angle FEH|$.

SOLUTION.

The Additivity Postulate implies the equations

$$|\angle ABC| \;=\; |\angle ABG| + |\angle CBG|\,, \qquad |\angle DEF| \;=\; |\angle DEH| + |\angle HEF|$$

and we can rewrite these in the following forms:

$$|\angle ABC| - |\angle ABG| \;=\; |\angle CBG|\,, \qquad |\angle DEF| - |\angle DEH| \;=\; |\angle HEF|$$

By our assumptions the left hand sides of both equations are equal, and therefore the right hand sides must also be equal.

3. [*20 points*] Suppose that we are given a triangle $\Delta\, ABC$ and $D \in (BC)$ such that $d(B, A) = d(B, D)$. If $|\angle ABC| = 50^\circ$ and $|\angle ACB| = 30^\circ$, find $|\angle ADC|$. [*Hint:* Drawing a picture is probably worthwhile.]

SOLUTION. By the Supplement Postulate we know that $|\angle ADC| + |\angle ADB| = 180^\circ$, so if we can find $|\angle ADB$ then we can find $|\angle ADC|$. Since ΔADB is isosceles with legs $[BD]$ and $[BA]$, it follows that

$$180^\circ \quad = \quad |\angle ABC| \; + \; 2\,|\angle ADB| \quad = \quad 50^\circ \; + \; 2\,|\angle ADB|$$

so that $|\angle ADB| = 65^\circ$. If we substitute this into the original equation, we find that $|\angle ADC| = 180^\circ - 65^\circ = 115^\circ$.

4. [*15 points*] Given an ordered list four coplanar points A, B, C, D such that no three are collinear, state the conditions for these points (in the given order) to be the vertices of a convex quadrilateral.

SOLUTION.

If (A, B, C, D) is the ordered list of four points, then the conditions are that

(i) A and B lie on the same side of CD,

(ii) B and C lie on the same side of DA,

(iii) C and D lie on the same side of AB,

(iv) D and A lie on the same side of BC.

5. [*10 points*] Let L be the line in $\mathbb{R}^3$ which contains the points $(1,2,3)$ and $(4,5,6)$, and let M be the line through $(2,1,3)$ which is parallel to L. Find a second point on M.

SOLUTION.

The line L may be written as $(1,2,3)+W$, where W is spanned by the difference $(4,5,6)-(1,2,3) = (3,3,3)$. Therefore W is the set of all vectors of the form (t,t,t) for some real number t. It follows that the line M may be written as $(2,1,3)+W$, and consequently the second point can be any point of the form $(2+t, 1+t, 3+t)$, where $t \neq 0$.

6. [*20 points*] Suppose that we are given ΔABC such that $d(A,B) = 8$ and $d(B,C) = 17$, and we know that $d(A,C)$ is an integer which is a perfect square. Use the Triangle Inequality to show that $d(A,C)$ must be equal to 16.

SOLUTION.

The Triangle Inequality implies that

$$17 \quad = \quad d(B,C) \quad < \quad d(B,A) \; + \; d(A,C) \quad < \quad 8 \; + \; d(A,C)$$

so that $d(A,C) > 9$ and also

$$d(A,C) \quad < \quad d(A,B) \; + \; d(B,C) \quad = \quad 8 \; + \; 17 \quad = \quad 25$$

so that $d(A,C) < 25$. Sine the only perfect square between 9 and 25 is 16, it follows that $d(A,C) = 16$.