IV : Projective geometry

Projective geometry is all geometry.

Attributed to A. Cayley (1821 – 1895)

As its name might suggest, ***projective geometry*** deals with ***the geometric theory of projecting 3 – dimensional figures onto a flat (planar) surface***, but over the centuries it has become considerably more. Although the Cayley quotation almost certainly overstated the role of projective geometry even when it was supposedly made, it is true that ***the subject provides a unified framework for a wide range of classical topics, including all the material in this course*** (however, the development of this framework requires concepts well beyond the scope these notes). In the present section, we shall look at the original motivation for projective geometry and explain how it yields techniques for solving some geometric problems that are considerably more difficult to work by traditional methods from classical Greek geometry. Further information (often at a somewhat higher level) is given in various documents from the following online directory:

http://math.ucr.edu/~res/progeom

IV.1 : Perspective images

The senses delight in things duly proportioned, as in what is after their own kind.

Thomas Aquinas (1225 – 1274), ***Summa Theologica,*** Part 1, Question 5.

Let no one who is not a mathematician read my works.

Leonardo da Vinci (1452 – 1519), ***Trattato della Pittura*** (*Treatise on painting*)

It is often convenient to think of projective geometry as really beginning with the efforts of artists in the late Middle Ages to create pictures that were more "lifelike" in their representations of physical objects. This important shift in emphasis took place in Italian painting starting around the year 1300. Prior to that time the central objects of paintings were generally flat and more symbolic than real in appearance; frequently emphasis was on depicting religious or spiritual truths rather than the real, physical world. As society in Italy became more sophisticated, there was an increased interest in using art to depict a wider range of themes, some of which were physical rather than spiritual, and to so in a manner that more accurately captured the image that the human eye actually sees.

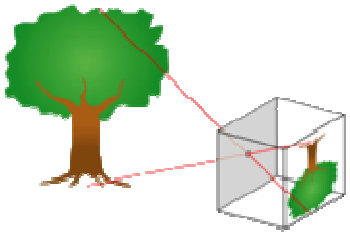
The earlier concepts of art are clearly represented in an illustration on the next page titled *Woman teaching geometry*, which appears Illustration at the beginning of a medieval translation of Euclid's ***Elements*** from around 1310.

(***Source:*** **http://en.wikipedia.org/wiki/Euclid's_Elements**)

In the picture, it looks as if the objects on the table are directly facing the viewer and ready to fall off the table's surface, and one can also notice the flat appearance of nearly all the faces.

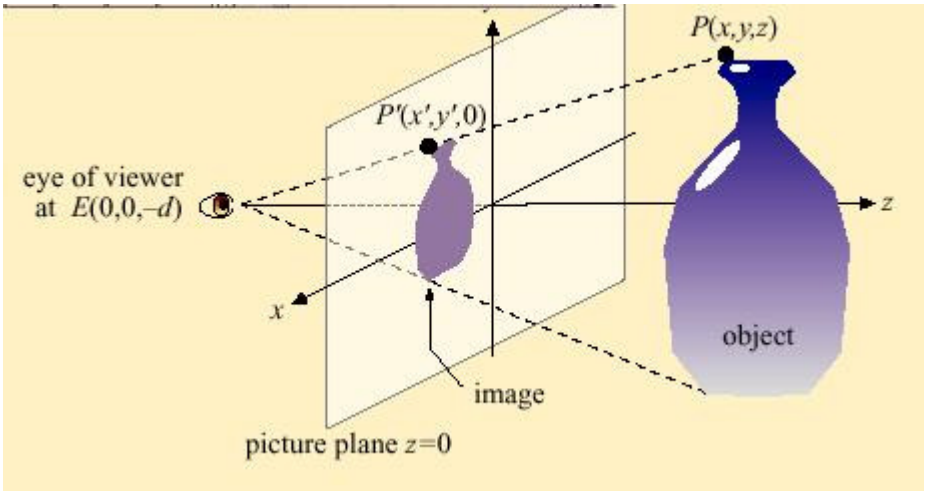
The paintings of Giotto (Ambrogio Bondone, 1267 – 1337), especially when compared to those of his predecessor Giovanni Cimabue (originally Cenni di Pepo, 1240 – 1302), show the growing interest in visual accuracy quite convincingly, and other painters from the 14th and early 15th century followed this trend. One can look at these paintings and see how the artists succeeded in showing things pretty accurately much of the time even though they were far from perfect. Eventually artists with particularly good backgrounds in Euclidean geometry began a systematic study of the whole subject and developed a mathematically precise theory of perspective drawing in the 15th and 16th centuries.

Before discussing the development of perspective drawing, we shall describe the main ideas. One way of deriving a mathematical model for drawing in perspective is to start with physical models. Since photographic images are good examples of perspective images, we shall start by analyzing a simple type of camera known as a ***pinhole camera,*** which is basically a rectangular box such that all but one of the sides are opaque, one is translucent (say made of waxed paper or ground glass), and there is a very small hole in the middle of the side opposite the translucent side. If such a camera is aimed at a physical object with the small opening facing the object, then a perspective image of the latter will be visible on the translucent side. As the drawing below suggests, this image will be inverted.



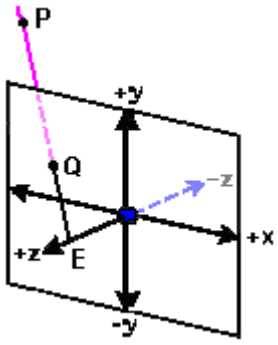
(***Source:*** **http://en.wikipedia.org/wiki/Pinhole_camera**)

The illustration of the pinhole camera leads to a simple rule for finding perspective images. Specifically, if one take the line joining the center of the small opening to a point on the object (in this case a tree), then the perspective image of the latter point is where that line meets the plane of the translucent "screen." Another possible model is given by placing a flat, transparent piece of glass between the observer's eye and the object, where the plane of the glass is perpendicular to the line passing through the eye and directed straight ahead.



(***Source:*** **http://www.cs.appstate.edu/~sjg/class/1010/wc/geom/perspective/**)

Formally, we may proceed as follows: Assume that the eye **E** is at some point on the positive $z-$ axis, the canvas or photographic plate is the $xy-$ plane, and the object **P** to be included in the painting is at the point **P,** which is on the opposite side of the the $xy-$ plane as the eye **E.** Then the image point **Q** on the canvas will be the point where the line **EP** meets the $xy-$ plane.



A detailed analysis of this geometrically defined mapping yields numerous facts that are logical consequences of the construction and Euclidean geometry. For example, one immediately has the following conclusion:

Proposition 1. *If* **P, P′** *and* **P″** *are collinear points on the opposite side of* **E** *and* **Q, Q′** *and* **Q″** *are their perspective images, then* **Q, Q′** *and* **Q″** *are also collinear.*

Proof. Let **L** be the line containing the three points. Then there is a plane **A** containing **L** and the point **E;** the three points **Q, Q′** and **Q″** all lie on the

intersection of the plane **A** with the xy – plane. Since the intersection of two planes is a line it follows that the three points must lie on this line.■

One significant difference between the mathematical and physical models is worth noting at this point. For drawing purposes, the only points which matter are those which are "in front of **E**." In other words, if we let $\mathcal{F}$ denote the "face plane" through **E** which is parallel to the xy – plane, then these are the points which lie on the same side of $\mathcal{F}$ as the xy – plane itself. However, one can also define the perspective images of points which are on the opposite side of $\mathcal{F}$ by the geometric construction given above, and we shall do so from now on. Of course, there are also many other things about perspective projections that are important, and we shall discuss some of them in later sections of this unit.

It is enlightening to examine some paintings from the 14th and 15th centuries to see how well they conform to these rules for perspective drawing. In particular, the site

http://www.ski.org/CWTyler_lab/CWTyler/Art%20Investigations/PerspectiveHistory/Perspective.BriefHistory.html

analyzes Giotto's painting ***Jesus Before the Caïf*** (1305) and shows that the rules of perspective are followed very accurately in some parts of the painting but less accurately in others.

Historical notes. We know that some aspects of perspective drawing were known in ancient times, and the concept appears explicitly in the writings of Vitruvius (Marcus Vitruvius Pollio, $c.$ 80 B. C. E. – 25 B. C. E.). However, the first medieval artist known to investigate the geometric theory of perspective as a subject in its own right was Fillipo Brunelleschi (1377 – 1446), and the first text on the theory was *Della Pittura*, which was written by Leon Battista Alberti (1404 – 1472). The influence of geometric perspective theory on paintings during the fifteenth century is obvious upon examining works of that period. The most mathematical of all the works on perspective written by the Italian Renaissance artists in the middle of the 15th century was *On perspective for painting (De prospectiva pingendi)* by Piero della Francesca (1412 – 1492). Not surprisingly, there were many further books written on the subject at the time (including a lost work by Leonardo da Vinci), of which we shall only mention the *Treatise on Mensuration with the Compass and Ruler in Lines, Planes, and Whole Bodies*, which was written by Albrecht Dürer (1471 – 1528) in 1525.

Here are some additional online references for the theory of perspective along with a few examples:

http://mathforum.org/sum95/math_and/perspective/perspect.html

http://www.math.utah.edu/~treiberg/Perspect/Perspect.htm

http://www.dartmouth.edu/~matc/math5.geometry/unit11/unit11.html

Here is a link to a perspective graphic that is animated:

http://gaetan.bugeaud.free.fr/pcent.htm

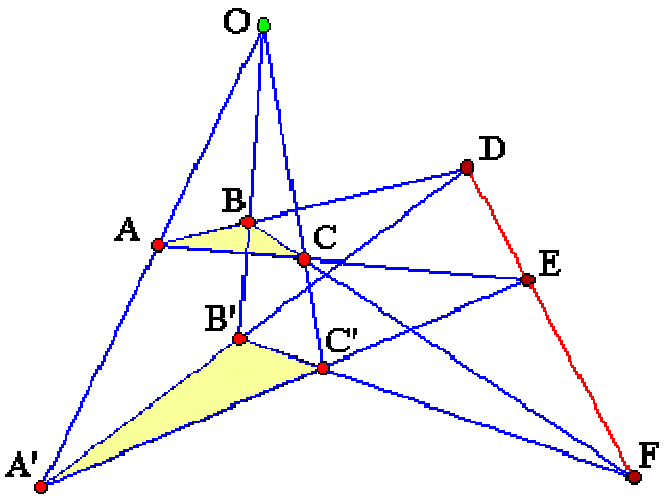
One can use the theory of perspective to determine precisely how much smaller the image of an object becomes as it recedes from the xy – plane, and more generally one can use algebraic and geometric methods to obtain fairly complete quantitative information about the perspective image of an object. Such questions can be answered very systematically and efficiently using computers, and most of the time (if not always)

the $3-$ dimensional graphic images on computer screens are essentially determined by applying the rules for perspective drawing systematically. We shall not give explicit formulas here, but we shall discuss some aspects of such formulas later in this unit.

From perspective drawing to projective geometry

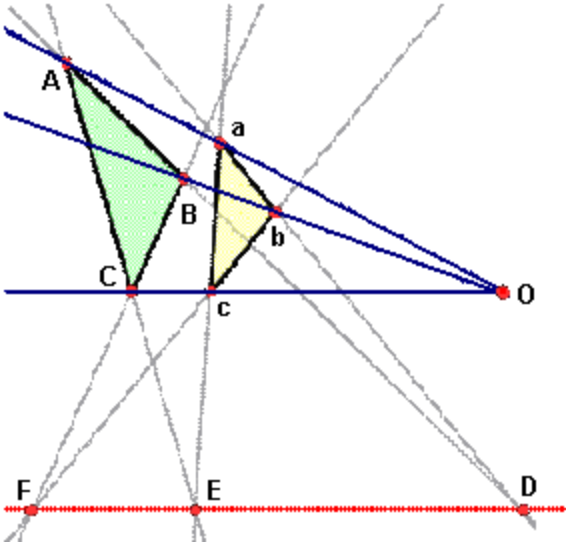
The mathematical theory of perspective drawing originated as a practical application of classical geometry, but it was also the first step in the development of new ways of analyzing certain geometrical problems, many (but not all) of which were much different from those which were studied in classical Greek geometry. Eventually mathematicians began to realize that they could use perspective projections to discover new geometrical facts. We conclude this section with a fundamental example which illustrates how ideas involving geometric perspective evolved into the beginnings of projective geometry.

Suppose that we have two distinct triangles $\triangle$**ABC** and $\triangle$**A′B′C′** such that the lines **AA′**, **BB′** and **CC′** joining the corresponding vertices all meet at some point **O**. Suppose further that the lines containing the corresponding sides intersect; more precisely, suppose that the lines **AB** and **A′B′** meet at **D**, the lines **AC** and **A′C′** meet at **E**, and the lines **BC** and **B′C′** meet at **F**. The picture below suggests that the points **D**, **E**, and **F** all lie on a straight line.



(***Source:*** **http://www.jimloy.com/geometry/desargue.htm**)

The first reaction to such a picture is to ask whether this is just a coincidence, and the obvious way to test this is to look at a different picture in which all the same conditions are satisfied. One example from **http://en.wikipedia.org/wiki/Desargues'_theorem** is depicted below.



The interactive figure in **http://www.pandd.demon.nl/cabrijava/transv_des.htm** strongly suggests that the three points **D, E,** and **F** are always collinear (unfortunately, the text is in Dutch; if one clicks and drags on any of the points **A, B, C** or **A′, B′, C′** then the positions of the points **D, E, F** will change but these points will still be collinear). In fact, one can prove that under the stated conditions the points **D, E,** and **F** are ***always*** collinear, and this is known as ***Desargues' Theorem*** because of the crucial role that G. Desargues (1591 – 1661) played in its discovery.

There are two significant points to note about this theorem. First of all, neither the hypothesis nor the conclusion has anything to do with measurements. This leads directly to the second point: By themselves, the classical theorems in the ***Elements*** do not provide much insight into how one might prove the result; with the tools developed thus far in this course, the most promising approach would probably involve using the Theorem of Menelaus in the exercises for Section **I.4,** but even this would require some work. Furthermore, even if one can put together such a proof (and this is possible), there is still an obvious question of motivating the result and understanding how it may have been originally discovered. Since this section of the notes deals with perspective projections, one might guess that the latter were somehow involved.

One way of thinking about Desargues' Theorem is to interpret a planar configuration in the theorem as a perspective projection (or photographic image) of a **3** – dimensional figure in which the points **Q, X, Y** and **Z** are not coplanar and these points project to **O, A, B** and **C** respectively. It follows that the points **X′, Y′** and **Z′** are also not collinear and the planes of **△XYZ** and **△X′Y′Z′** are distinct (we assume that **X′, Y′** and **Z′** project to **AA′, BB′** and **CC′** respectively). If we know that the lines **XY** and **X′Y′** meet at **U,** the lines **XZ** and **X′Z′** meet at **V,** and the lines **YZ** and **Y′W′** meet at **W,** then we know that **U, V** and **W** project to **D, E** and **F** respectively. Now the points **U, V** and **W** lie on plane **XYZ** and also on plane **X′Y′Z′.** By construction these planes are distinct, so this intersection must be a line. Finally, the image of this line under perspective projection is a line, and it contains the images of **U, V** and **W,** which are just the original points **D, E** and **F.** This implies that the latter three points are collinear. — Note that ***this is* NOT *a complete proof*** because we have not tried to verify that our original configuration actually is the image of some **3** –

dimensional configuration, but **our purpose here was to motivate the result rather than to prove it**; we shall give a rigorous proof in Section **4** of this unit. However, we should note that it is possible to prove that one can realize a planar configuration of the given type as a perspective projection of a **3** – dimensional figure (although we shall not verify this fact in the notes).

Desargues also made several other discoveries that are now fundamental to projective geometry, and many will be discussed in later sections of this unit. Although there were subsequent contributions to the subject by B. Pascal (1623 – 1662) and P. de la Hire (1640 –1718), the subject did not develop much further until the early 19th century with the work of J. – V. Poncelet (1788 – 1867) and J. Gergonne (1771 – 1859). We shall mention some important advances by Poncelet and Gergonne at various points in this unit, and we shall discuss Pascal's Theorem in Section **5** of this unit. The revival of interest in projective geometry and perspective transformations is also reflected by somewhat different work of G. Monge (1745 – 1818) on the systematic application of mathematics to constructing planar images of objects in space; his theory of ***descriptive geometry*** evolved into the theory of ***orthographic projection***, which is a fundamental graphical technique in modern mechanical drawing. Here are some online references for orthographic projection:

http://en.wikipedia.org/wiki/Orthographic_projection

http://www.geneng.mtu.edu/courses/3000/current/07.spac.vis2.pdf

http://mathworld.wolfram.com/OrthographicProjection.html

IV.2 : Adjoining ideal points

> Extending the space ... [is a] fruitful method for extracting understandable results from the bewildering chaos of special cases.
>
> J. Dieudonné

In calculus — and particularly in the theory of limits — it is frequently convenient to add one or two ***numbers at infinity*** to the real number system. Among the reasons for this are the following:

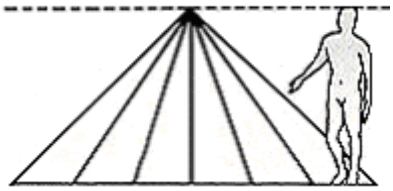
1. It allows one to formulate otherwise complicated notions more understandably (for example, infinite limits).
2. It emphasizes the similarities between the infinite limit concept and the ordinary limit concept.
3. It allows one to perform some formal manipulations with limits much more easily.

For example, suppose we add a single point at infinity to the real numbers; as usual, we denote it by the symbol ∞. If f is a real valued rational function of the form $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials such that $q(x)$ is nonzero, then strictly speaking f is not definable at the finite set of real roots of the denominator.

However, an inspection of the graph of f suggests that one define its value at these roots to be ∞. Proceeding further along these lines, one can even define $f(\infty)$ so that f is "continuous at ∞," where the value may be an ordinary real number or ∞.

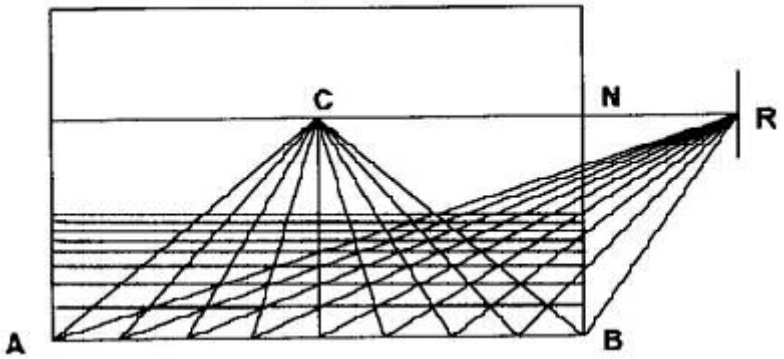
The purpose of this section is to discuss a similar phenomenon in projective geometry. We shall start with a basic observation about perspective transformations.

VANISHING POINT PROPERTY. *If* **L, M** *and* **N** *are mutually parallel lines, then their perspective images pass through a single point (the* ***vanishing point****) on the image plane. The set of all vamishing points on all lines in a given plane is collinear, and the line is called the* ***vanishing line.*** *Furthermore, as an object moves away from the viewer along one of these lines, the limiting position of the object will be this vanishing point.* ■



(***Source:*** http://www.collegeahuntsic.qc.ca/Pagesdept/Hist_geo/Atelier/Parcours/Moderne/perspective.html)

As the drawing above suggests, the vanishing line corresponds to the visual horizon line in a picture. Here is another drawing depicting two different families of mutually parallel lines which have different vanishing points on the vanishing line.



(***Source:*** http://www.math.nus.edu.sg/aslaksen/projects/perspective/alberti.htm)

The preceding drawings suggest that we might want to adjoin a points at infinity to each line such that **(1)** all lines parallel to a given line **L** have the same point at infinity as the line **L,** **(2)** the perspective images of these points of infinity lie on the horizon line.

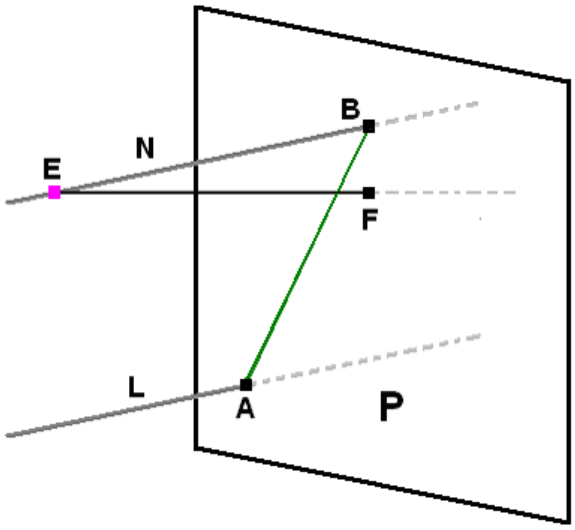
Some mathematical justifications

Before elaborating on these points, for the sake of completeness, we shall prove that our observations on perspective projections are mathematically valid. A reader who wishes to skip the statement and proof of the next results may do so without loss of continuity.

Theorem 1. *Let* **P** *be a plane, let* **E** *be a point in space not on* **P,** *and let* **L** *be a line in space which is not parallel to* **P.** *Take* **N** *to be the unique line through* **E** *which is equal or parallel to* **L,** *and let* **B** *be the point at which* **N** *meets* **P.**

1. *If* $\mathbf{L}^*$ *is the line in* **P** *which contains the image of* **L** *under perspective projection onto* **P** *with center* **E,** *then* $\mathbf{L}^*$ *contains* **B.**
2. *If* **M** *is an arbitrary line which is parallel to* **L** *and* $\mathbf{M}^*$ *is the line in* **P** *which contains the image of* **L** *under perspective projection onto* **P** *with center* **E,** *then either the lines* $\mathbf{L}^*$ *and* $\mathbf{M}^*$ *are equal or they meet at* **B.**
3. *Suppose that* **Q** *is a plane which is not parallel to* **P,** *and suppose that the line* **L** *is contained in* **Q.** *If* **Q′** *is the unique plane through* **E** *which is equal or parallel to* **Q,** *then* **P** *meets* **Q′** *in a line* **H,** *and the intersection* $\mathbf{L} \cap \mathbf{H}$ *contains* **B.**

In everyday language, **H** is the ***horizon*** or ***vanishing line*** of the plane **Q** for the perspective projection onto **P;** in physical terms, if an object **X** travels away from **P** along the line **L,** then **B** is the limiting position for the image $\mathbf{X}^*$ on **P** as **X** goes off to infinity.



Proof. As suggested by the diagram, we take **A** and **B** to be the intersections of **L** and **N** with the plane **P** (these exist because the lines are not parallel to **P**). The point **F** represents the foot of the perpendicular from **E** to **P.**

We start by proving the first assertion in the conclusion of the theorem. If $\mathbf{L} = \mathbf{N}$ then the conclusion is trivial, so assume that $\mathbf{L} \parallel \mathbf{M}$. By assumption we know that there is a plane, say **LN,** containing both **L** and **N.** Since **LN** and **P** have two points **A** and **B** in common (and $\mathbf{A} \neq \mathbf{B}$ because these points lie on parallel lines), it follows that **LN** meets **P** in the line **AB.** We claim that **AB** is the line containing the image of **L** under perspective projection. Let **X** be a typical point of **Y** for which the perspective projection is defined, and let **Y** be its image. Then **Y** lies on **P** and on the line **EX.** Since **X** and **E** lie on the plane **LN,** the entire line **EX** lies in **LN;** in particular, **Y** lies on **LN.** But this means that **Y** lies in $\mathbf{LN} \cap \mathbf{P} = \mathbf{AB}$.

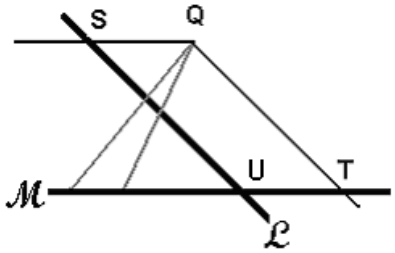
The second part follows immediately from the first, for in this case one finds that the image **M*** will be a line containing **B.**

Finally, to prove the third part, we start by proving that **N** is contained in **Q′.** Linear algebra provides an easy way of doing so. Since **L** and **N** are equal or parallel, it follows that we can write **N = z + L** for some vector **z.** Using this, we write **E = z + v** for some point **v** on **L;** since **Q** and **Q′** are equal and parallel such that the first contains **L** (hence also **v**) and the second contains **E,** it follows from the Coset Property that **Q′ = z + Q.** Therefore we have that **N = z + L** is contained in **z + Q = Q′.**

Since **P** is not parallel to **Q** and **Q′** is parallel or equal to **Q,** it follows that **Q′** is not parallel to **P** and hence meets it in a line **H.** Since **N** is contained in **Q′,** it follows that the intersection **N ∩ P,** which consists of only the point **B,** will be contained in **H = Q′ ∩ P.■**

We shall now give some provisional justification for the remarks about continuity. For this purpose it is convenient to look at a stripped down situation in which we examine the perspective image of one line upon another in the same plane. Actually, there are two cases depending upon whether or not the lines are parallel, but the latter can be handled very easily and does not shed any insight upon our discussion, so an analysis of these cases is left to the exercises.

Since we are interested in two lines, say $\mathcal{L}$ and $\mathcal{M}$, which are not parallel, they will meet at some point that we shall call **U.** In this case perspective projection comes close to defining a $1-1$ correspondence between the points of $\mathcal{L}$ and the points of $\mathcal{M}$, but this is not quite the case. Suppose that **S** is a point on $\mathcal{L}$ such that **QS** || $\mathcal{M}$. Then it is not possible to define the perspective projection of **S** onto $\mathcal{M}$. Similarly, if **T** is a point on $\mathcal{M}$ such that **QT** is parallel to $\mathcal{L}$, then **T** is not the perspective projection of any point on $\mathcal{L}$. In fact, what happens is that perspective projection defines a $1-1$ correspondence from all the points of $\mathcal{L}$ except **S** to all the points of $\mathcal{M}$ except **T.**



Our previous remarks suggest that if we add points at infinity to the two lines, then **S** should go to the point at infinity for $\mathcal{M}$ and the point at infinity for $\mathcal{L}$ should go to **T.** If we do this, then we have a $1-1$ correspondence of the extended lines $\mathcal{L}^*$ and $\mathcal{M}^*$ (where each is obtained from the original line by adding a point at infinity). Set theory allows us to define such functions, but we would also like to know that this definition has the sorts of continuity properties we claimed earlier in this section.

Consider what happens if we look at points approaching **S** or **T.** If we let $\mathbf{X} \in \mathcal{L}$ approach **S,** then the distance from its perspective image **Y** and the point **T** appears to go off to infinity. Similarly, if we let $\mathbf{Y} \in \mathcal{M}$ approach **T** and suppose that **X** is the point on $\mathcal{L}$ whose perspective image is **Y,** then the distance between **X** and **S** also appears to go off to infinity.

We must now show that if we write things down explicitly in terms of linear algebra and barycentric coordinates, then ***we can make all of this mathematically precise.*** The first step is to note that **Q, T, U** and **S** (in that order) form the vertices of a parallelogram; it follows that we have the vector equation $\mathbf{Q} = \mathbf{S} + \mathbf{T} - \mathbf{U}$.

Suppose now that $\mathbf{X} \in \mathcal{L}$ and $\mathbf{Y} \in \mathcal{M}$ correspond under the perspective projection. Then we can find scalars p and q such that $\mathbf{X} = \mathbf{U} + p(\mathbf{S} - \mathbf{U})$ and $\mathbf{Y} = \mathbf{U} + q(\mathbf{T} - \mathbf{U})$; we need to determine the relationship between p and q, and the key to doing so is to recall that **Y** lies on the line joining **Q** and **X,** so that $\mathbf{Y} = \mathbf{Q} + z(\mathbf{X} - \mathbf{Q})$ for some scalar z. If we expand **X** and **Q** in terms of **S, T** and **U** using the formulas of this paragraph, then after grouping like terms we find that

$$\mathbf{Y} = (1 + pz - z)\mathbf{S} + (1 - z)\mathbf{T} + (2z - pz - 1)\mathbf{U}$$

and the sum of the coefficients is 1. Since we also have $\mathbf{Y} = q\mathbf{T} + (1 - q)\mathbf{U}$, we may set the corresponding barycentric coordinates equal and consequently obtain the equations

$$1 + pz - z = 0 \qquad \text{and} \qquad 1 - z = q.$$

Eliminating z from these equations, we obtain the relation

$$q = \frac{p}{p-1} = 1 - \frac{1}{p-1}$$

which shows that if **X** approaches **S,** so that $p \to 1$, then $q \to \infty$, and therefore **Y** goes off to infinity. Similarly, the relation shows that if **X** goes off to infinity (in *either* direction!), so that $p \to \infty$, then $q \to 1$, and therefore **Y** approaches **T.** In particular, the latter reflects our assertion that the vanishing point on the image line is the limiting position of the image of an object on the original line as the object goes off to infinity. ■

Incidence structure of the projective extensions

> One reason for the invention of the projective plane was to simplify the incidence geometry of the Euclidean plane.
>
> Ryan, p. 127

For $n = 2$ or 3, the preceding discussion suggests the construction of new sets of points $\mathbb{P}(\mathbb{R}^n)$ which have ***ordinary points*** consisting of the points of $\mathbb{R}^n$ and a new class of ***ideal points,*** with one for every maximal family of pairwise parallel lines in $\mathbb{R}^n$

(by this we mean the family of all lines parallel or equal to a given line **L**). We shall conclude this section by defining lines and (if $n = 3$) planes in $\mathbb{P}(\mathbb{R}^n)$ which are compatible with the usual notions for ordinary points in $\mathbb{R}^n$.

There are two classes of lines.

1. For each line **L** in $\mathbb{R}^n$, an ***extended ordinary line*** $\mathbb{P}$**(L)** which consists of the points on **L** and the ***ideal point*** $\mathbf{L}^+$ given by the unique maximal family of mutually parallel lines containing **L**.
2. For each plane **P**, an ***ideal line*** (or ***line at infinity***) $\mathbb{L}\infty$**(P)** which consists of the ideal points corresponding to all the ordinary lines which are contained in **P**; if $n = 2$, this means there is just one ideal line and we call it $\mathbb{L}\infty$.

Furthermore, if $n = 3$ then there are two similar classes of planes.

1. For each plane **Q** in $\mathbb{R}^3$, an ***extended ordinary plane*** $\mathbb{P}$**(Q)** which consists of the points on **Q** and the points on the ideal line $\mathbb{L}\infty$ **(Q)** described above.
2. There is also an additional ***ideal plane*** (or ***plane at infinity***) $\mathbb{P}\infty$ consisting of all the ideal points.

If we adopt these notions of lines and planes for $\mathbb{P}(\mathbb{R}^2)$ and $\mathbb{P}(\mathbb{R}^3)$ then one obtains systems which satisfy the corresponding incidence axioms from Section **II.1** of these notes. We shall not give the detailed verification here, but we shall note two important additional properties which are satisfied by these systems and are quite different from the properties of Euclidean lines and planes.

Proposition 2. *Two coplanar lines in* $\mathbb{P}(\mathbb{R}^3)$ *always have a point in common, and two planes in* $\mathbb{P}(\mathbb{R}^2)$ *always have a line in common.*

We shall prove these facts in the next section of the notes.

Ideal points and Galilean transformations

In the discussion of perspective transformations, we have seen that ideal points provide an effective means for stating relatively complicated conclusions in a more unified manner. In particular, we have noted that a perspective projection from a given line $\mathcal{L}$ to a second, nonparallel, line $\mathcal{M}$ determines a $1-1$ correspondence from the extended line $\mathcal{L}^*$ to the extended line $\mathcal{M}^*$. There are numerous other instances in geometry where statements involving a list of special cases can be reformulated in a unified manner by use of ideal points. We shall discuss several examples of this phenomenon in Section **5** of this unit, and we shall conclude this section with yet another example of this sort.

Theorem 3. *Let* **T** *be an isometry of* $\mathbb{R}^2$ *given by a Galilean transformation*

$$\mathbf{T(x) = Ax + b}$$

where **A** *is given by an orthogonal matrix with a* **positive** *determinant, and let* **S** *be a subset of* $\mathbb{R}^2$ *such that* $\mathbf{T(x) \neq x}$ *for all* $\mathbf{x \in S}$. *Then either* **(1)** *the perpendicular bisectors for all closed segments* **[xT(x)]** (*where* $\mathbf{x \in S}$) *all pass through a single point, or else* **(2)** *these perpendicular bisectors are parallel or equal to one of the lines in the family.*

This result is stated and proved on pages 355 – 356 of the book by *Usiskin, Peressini, Marchisotto and Stanley* that was mentioned earlier in these notes. Our interests here are limited to restating the conclusion using ideal points. In the language of this section, the conclusion states that the extended perpendicular bisectors (obtained by adjoining points at infinity) either **(1)** pass through a single ordinary point, or else **(2)** pass through a single ideal point. Of course, one can restate this conclusion in a unified form because both options mean that the extended perpendicular bisectors all pass through a single point (which may be either ordinary or ideal).

Incidentally, there is a companion result to the theorem above if $\mathbf{T(x) = Ax + b}$ is a Galilean transformations for which **A** is an orthogonal matrix whose determinant is **negative**; the statement and proof are given on page **356** of the book cited above.

Ideal points — fact or fantasy?

The concept of ***ideal points*** or points at infinity may seem extremely unnatural because it is not physically possible to visualize them. However, such points are abstract mathematical objects rather than concrete physical objects, and the following passage from an old encyclopedia article explains why mathematicians are comfortable with using them.

> Ordinary points [*whose physical dimensions are all equal to zero*] are just as much idealized as the points at infinity. No one has ever seen an actual point or realized it by an experiment of any sort. Like the point at infinity it is an ideal [abstract] creation which is useful for some of the purposes of science.
>
> [***Encyclopædia Britannica***, 14th Ed. (1956), Vol. 18, p. 573, article on Projective Geometry]

The final clause in this citation is important enough to merit further comment; namely, the rules of logic guarantee that one can formally introduce ideal points into geometry without creating logical problems, but the usefulness of ideal points for effectively studying questions of independent interest provides the crucial motivation for doing so. Here is another quotation [***Source***: N. Altshiller–Court, ***Mathematics in Fun and in Earnest***, pp. 110 – 112.]:

> [With the introduction of points at infinity,] The propositions of projective geometry acquire a simplicity and a generality that they could not otherwise have. Moreover, the elements at infinity give to projective geometry a degree of unification that greatly facilitates the thinking in this domain and offers a

> suggestive imagery that is very helpful **...** On the other hand, projective geometry stands ready to abandon these **...** whenever that seems desirable, and to express the corresponding propositions in terms of direction of a line **...** to the great benefit of **...** geometry. **...** The extra point which projective geometry claims to add to the Euclidean line is [merely] the way in which projective geometry accounts for the property of a straight line which Euclidean geometry recognizes as the "direction" of the line. The difference between the Euclidean line and the projective line is purely verbal. The geometric content is the same. **...** Such a change in nomenclature does not constitute an [actual] increase in the geometric content.

Of course, one goal for the rest of this unit is to describe some uses of ideal points to study "real" geometrical questions. However, this will require the development of some additional algebraic tools and logical concepts in the next two sections. The geometrical applications themselves will be presented in Sections **5** and **6** of this unit.

IV.3 : Homogeneous coordinates

> Generations of mathematicians are growing up who are on the whole splendidly trained, but suddenly find that, after all, they do need to know what a projective plane is.
>
> I. Kaplansky (1917 – 2006)

During much of the 19th century there were sharp differences of opinions among some mathematicians about the best approach to projective geometry, with some favoring a strictly synthetic approach similar to that of the ***Elements*** and others advocating the introduction and systematic use of coordinates. Eventually there was a consensus that both approaches were important to the subject, with the strengths of one often compensating for the disadvantages of the other. The following quotation by Lagrange predates the 19th century activity in projective geometry, but it reflects the situation for that subject quite well.

> Even though analysis [coordinate or linear – algebraic methods] may have advantages over the old geometric methods **...** there are nevertheless problems in which the latter appear more advantageous.

In the previous section we noted that some geometric concepts and computations can be greatly simplified if one has suitable points at infinity, and we discussed the process of adjoining such points from a largely synthetic approach. In this section we shall discuss the process from an algebraic approach. It turns out that many geometric computations involving points at infinity can be greatly simplified with the introduction of suitable coordinates, and they will be used repeatedly throughout the rest of this unit.

Clearly any theories of numerical coordinates for $\mathbb{P}(\mathbb{R}^2)$ and $\mathbb{P}(\mathbb{R}^3)$ will be particularly useful if they are closely related to data from ordinary coordinate geometry. In particular, here are two fairly reasonable criteria we would like to have:

1. Projective coordinates for ***ordinary points*** should have some close and obvious relation to the usual rectangular coordinates.

2. Projective coordinates for ***ideal points on ordinary lines*** should have some close and obvious relation to sets of direction numbers for the line.

Our definition of ***direction numbers for a line*** is exactly the same as in elementary courses. If two points **a** and **b** lie on the line **L** in $\mathbb{R}^n$, then the ordinary coordinates of $\mathbf{b} - \mathbf{a}$ are said to be ***a set of direction numbers for*** the line **L**; since there are many choices for the points **a** and **b**, it is clear that sets of direction numbers for a line are not unique, but if **u** and **v** are vectors giving direction numbers for a line then these vectors are nonzero multiples of each other (**PROOF:** Write the line as $\mathbf{z} + \mathbf{W}$ for some $1 -$ dimensional vector space **W**, and notice that if a and b are two points on **L**, then $\mathbf{b} - \mathbf{a}$ is a nonzero vector in **W**; since any two nonzero vectors in **W** are nonzero multiples of each other, the validity of the assertion follows).

It is also important to mention two types of flexibility that we have in describing projective coordinates.

3. Two or more lists of coordinate values may determine the same point.
4. It is sometimes convenient to use more than n coordinates to specify a point in an $n -$ dimensional object.

Spherical and rectangular coordinates for the standard unit sphere in $\mathbb{R}^3$ are excellent illustrations of these principles. Spherical coordinates correspond to latitude and longitude, at least after a simple change of variables, and for many purposes it is useful to have two ways of writing a point. For example, one gets the north and south poles if the latitude is 90 degrees regardless of what value one uses for the longitude. Similarly, if one is going eastbound around the equator, in some contexts it might be better to say one is at 181 degrees east of Greenwich rather than 179 degrees west of it. Although the three rectangular coordinates for a point on the surface of a $2 -$ dimensional sphere carry some redundant information, it is much easier to study some of the symmetry properties of the sphere using rectangular rather than spherical coordinates (in contrast, latitudes and longitudes do not play symmetrical roles in specifying a point on the sphere; lines of constant longitude lie on great circles but all lines of constant latitude except for the equator are not great circles). Of course, barycentric coordinates for a point in $\mathbb{R}^3$ are another extremely useful example of the fourth criterion.

We shall start with the fourth criterion for constructing coordinates. In order to specify a point in $\mathbb{P}(\mathbb{R}^n)$ we shall use $n + 1$ coordinates. Given an ordinary line **L** in $\mathbb{R}^n$, suppose that a set of direction numbers for it is given by some nonzero vector **v** in $\mathbb{R}^n$. Then the vector $(\mathbf{v}, 0) \in \mathbb{R}^{n+1}$ will be defined to be a set of homogenous coordinates for the point at infinity $\mathbf{L}^+$. Since two nonzero vectors **v** and **w** are direction numbers for the same line if and only if one vector is a nonzero multiple of each other, it follows that for each $c \neq 0$ the vectors $(c\mathbf{v}, 0)$ are also homogeneous coordinates for the point at infinity on the given line.

Similarly, if $\mathbf{x}$ is a point in $\mathbb{R}^n$, then we take $(\mathbf{x}, 1) \in \mathbb{R}^{n+1}$ to be one set of projective coordinates for the corresponding ordinary point in $\mathbb{P}(\mathbb{R}^n)$, and to preserve uniformity with the previous paragraph also take every nonzero multiple $(c\mathbf{v}, c)$.
We now note two crucial properties of these definitions.

Proposition 1. *Every nonzero vector in $\mathbb{R}^{n+1}$ is a set of projective coordinates for some point in $\mathbb{P}(\mathbb{R}^n)$, and two nonzero vectors in $\mathbb{R}^{n+1}$ are projective coordinates for the same point if and only if one is a nonzero multiple of the other.*

Notation. Because of the second clause in the proposition, the projective coordinates described above are almost universally called ***homogeneous coordinates*** for a point in the projective space $\mathbb{P}(\mathbb{R}^n)$.

Proof. Suppose we are given $(\mathbf{z}, t) \in \mathbb{R}^{n+1}$, where $\mathbf{z}$ is a vector in $\mathbb{R}^n$ and t is a scalar such that either $\mathbf{z}$ or t is nonzero. There are two cases depending upon whether t is nonzero. If it is nonzero, then $(\mathbf{z}, t)$ represents the ordinary point $t^{-1}\mathbf{z}$, while if $t = 0$ then $(\mathbf{z}, 0)$ represents the point at infinity on the ordinary line joining the (vector) origin $\mathbf{0}$ to $\mathbf{z}$ (recall that the latter is nonzero).

By definition, if two vectors in $\mathbb{R}^{n+1}$ are nonzero multiples of each other, then they represent the same point in $\mathbb{P}(\mathbb{R}^n)$. Conversely, suppose that the two nonzero vectors $(\mathbf{z}, t)$ and $(\mathbf{w}, u)$ represent the same point in $\mathbb{P}(\mathbb{R}^n)$. Since the coordinates for ordinary points have nonzero last entries and the coordinates for ideal points have zero in their last entry, it follows that t and u are either both equal to zero or both not equal to zero. In the second case it follows that the ordinary point represented by the two nonzero vectors is given by $t^{-1}\mathbf{z} = u^{-1}\mathbf{w}$, so that $\mathbf{z} = tu^{-1}\mathbf{w}$ and hence we have $(\mathbf{z}, t) = tu^{-1}(\mathbf{w}, u)$. We now turn to the first case, in which the final entries are both zero. By the previous paragraph, we know that $(\mathbf{z}, 0)$ and $(\mathbf{w}, 0)$ represent the point at infinity on the ordinary lines joining $\mathbf{0}$ to $\mathbf{z}$ and $\mathbf{0}$ to $\mathbf{w}$, and therefore it follows that these two lines are either parallel or equal. Since the origin $\mathbf{0}$ belongs to both, it follows that they must be equal, which in turn implies that $\mathbf{w}$ must lie in the $1-$ dimensional vector subspace of $\mathbb{R}^n$ spanned by $\mathbf{z}$; the latter in turn means that $\mathbf{w}$ and $\mathbf{z}$ are nonzero multiples of each other, so that the same is also true for both $(\mathbf{z}, 0)$ and $(\mathbf{w}, 0)$.■

Corollary 2. *For $n = 2$ or 3, there is a $1-1$ correspondence between the points of $\mathbb{P}(\mathbb{R}^n)$ and $1-$ dimensional vector subspaces of $\mathbb{R}^{n+1}$.*■

In Unit **I** we showed that lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$ have relatively straightforward interpretations in terms of linear algebra, and our next objective is to show that lines and planes in $\mathbb{P}(\mathbb{R}^2)$ and $\mathbb{P}(\mathbb{R}^3)$ have similar interpretations. The first results parallel the standard descriptions of lines in $\mathbb{R}^2$ and planes in $\mathbb{R}^3$ as sets of points satisfying nontrivial linear equations.

Theorem 3. *A subset* Λ *of* $\mathbb{P}(\mathbb{R}^2)$ *is a line if and only if it is the set of points whose homogeneous coordinates satisfy a nontrivial linear homogeneous equation of the form* $a_1x_1 + a_2x_2 + a_3x_3 = 0$.

Note that if one set of homogeneous coordinates for a point satisfies the equation

$$a_1x_1 + a_2x_2 + a_3x_3 = 0$$

then ***all*** sets of homogenous coordinates satisfy the same equation.

Proof. Suppose we are given an ordinary line **L** and that it is definable by the nontrivial linear equation $Au + Bv + C = 0$ (nontriviality means at least one of A, B is nonzero). We **claim** that the extended line $\mathbb{P}$**(L)** is definable by the analogous homogeneous equation $Ax_1 + Bx_2 + Cx_3 = 0$. It follows immediately that the homogeneous coordinates for an ordinary point satisfy the second equation if and only if the point lies on **L**. Furthermore, we claim that the homogeneous coordinates of the ideal point $\mathbf{L}^+$ also satisfy the homogeneous equation. To see this, suppose that $\mathbf{P}_1 = (u_1, v_1)$ and $\mathbf{P}_2 = (u_2, v_2)$ are distinct ordinary points on **L**; then $\mathbf{P}_1 - \mathbf{P}_2$ gives a set of direction numbers for **L** and hence the ideal point $\mathbf{L}^+$ has homogeneous coordinates given by the vector $(u_1 - u_2, v_1 - v_2, 0)$. Since $\mathbf{P}_1$ and $\mathbf{P}_2$ lie on **L** we know that

$$Au_1 + Bv_1 + C = Au_2 + Bv_2 + C = 0$$

and if we subtract the second equation from the first we see that

$$A(u_1 - u_2) + B(v_1 - v_2) + C \cdot 0 = 0$$

so that the homogeneous coordinates of $\mathbf{L}^+$ satisfy the linear homogeneous equation in the claim. Now suppose that $\mathbf{M}^+$ is an arbitrary ideal point whose homogeneous coordinates satisfy this linear homogeneous equation, and suppose that homogeneous coordinates for $\mathbf{M}^+$ are given by $(z, w, 0)$. Then we have $Az + Bw = 0$, and by the basic results from linear algebra on solutions of homogeneous linear equations we know that must be a nonzero multiple of $(u_1 - u_2, v_1 - v_2, 0)$, so that both sets of homogeneous coordinates represent the same point and hence $\mathbf{L}^+ = \mathbf{M}^+$. This completes the proof for extensions of ordinary lines in $\mathbb{P}(\mathbb{R}^2)$.

The proof for the ideal line is much simpler, for ideal points are the only ones which have homogeneous coordinates of the form $(u, v, 0)$, and hence a point is an ideal point if and only if its homogeneous coordinates satisfy the equation $x_3 = 0$.

To see that every set of points whose homogeneous coordinates satisfy a nontrivial linear equation $a_1x_1 + a_2x_2 + a_3x_3 = 0$, note that **(1)** if either a_1 or a_2 is nonzero, then we have shown that set of points as above is the set of points on the extension of an ordinary line, **(2)** if both a_1 and a_2 are zero, then a_3 must be zero so that the equation has the form $a_3x_3 = 0$, and since this is equivalent to the equation $x_3 = 0$ we see that the set of points whose homogeneous coordinates satisfy the original equation is just the ideal line.■

Corollary 4. *A subset* Λ *of* $\mathbb{P}(\mathbb{R}^2)$ *is a line if and only if there is a* $2-$*dimensional vector subspace* **W** *of* $\mathbb{R}^3$ *such that a point* **X** *lies in* Λ *if and only if its sets of homogeneous coordinates are nonzero vectors of* **W**.

As before, if one set of homogeneous coordinates lies in a particular vector subspace of $\mathbb{R}^2$ and $\mathbb{R}^3$, then ***all*** sets of homogenous coordinates lie in this subspace.

Proof. We know that Λ is a line if and only if its homogeneous coordinates satisfy a nontrivial linear homogeneous equation of the form $a_1x_1 + a_2x_2 + a_3x_3 = 0$, and hence the points on Λ are those whose homogeneous coordinates are the nonzero vectors in the vector subspace **W** of solutions for that equation. Therefore, if Λ is a line then it satisfies the conclusion of the corollary. Conversely, if **W** is a $2-$dimensional vector subspaces of $\mathbb{R}^3$ then it is the set of solutions for some nontrivial linear homogeneous equation of the form and thus the set of points in $\mathbb{P}(\mathbb{R}^2)$ whose homogeneous coordinates lie in **W** are precisely the points on the line defined by the equation $a_1x_1 + a_2x_2 + a_3x_3 = 0$.■

There are analogous results for planes in $\mathbb{P}(\mathbb{R}^3)$, and the proofs are entirely similar.

Theorem 5. *A subset* Π *of* $\mathbb{P}(\mathbb{R}^3)$ *is a plane if and only if it is the set of points whose homogeneous coordinates satisfy a nontrivial linear homogeneous equation of the form* $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$.

Again, if one set of homogeneous coordinates for a point satisfies the equation

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$$

then ***all*** sets of homogenous coordinates satisfy the same equation.

Proof. We shall only indicate the points where the argument is not analogous to the proof of the previous theorem; these all involve extended ordinary planes for which at least one of the coefficients a_1, a_2, a_3 is nonzero. Once again we know that the ordinary points on the extended ordinary plane are simply those whose homogeneous coordinates satisfy the equation $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$, so we

need to check that the ideal points on this plane are precisely those whose homogeneous coordinates also satisfy this equation.

Let **Q** be the ordinary plane under consideration, and suppose it has the form $\mathbf{v} + \mathbf{W}$ for some vector **v** and some $2-$dimensional vector subspace **W.** Then the direction numbers for all lines joining two points in **Q** must lie in **W,** and conversely every nonzero vector **w** in **W** is a set of direction numbers for some line **M** in **Q;** specifically, one can take **M** to be the line joining **x** to $\mathbf{x} + \mathbf{w}$. By the same sort of argument used in the planar case, if we are given a line **L** in the ordinary plane **Q** and it has direction numbers given by (p_1, p_2, p_3), then we know that the direction numbers satisfy $a_1p_1 + a_2p_2 + a_3p_3 = 0$ and homogeneous coordinates for the ideal points $\mathbf{L}^+$ are given by $(p_1, p_2, p_3, 0)$.

Thus we have shown that the ideal points on the extended plane $\mathbb{P}(\mathbf{Q})$ all have homogeneous coordinates which satisfy the linear homogeneous equation we have associated to **Q.** In order to conclude the argument we need to show that if an ideal point has homogeneous coordinates satisfying the associated linear homogeneous equation $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$ then $\mathbf{M}^+$ lies in the extended plane $\mathbb{P}(\mathbf{Q})$. Now the set of solutions for $a_1x_1 + a_2x_2 + a_3x_3 = 0$ is a $2-$dimensional vector subspace of $\mathbb{R}^3$ which contains the vector subspace **W** described above, and therefore it must be equal to **W.** Therefore, if we are given an ideal point with homogeneous coordinates $(p_1, p_2, p_3, 0)$ which satisfy the basic linear homogeneous equation under consideration, then we know that (p_1, p_2, p_3) must lie in **W,** which means that this $3-$dimensional vector must give the direction numbers for some line **N** in **Q,** so that $\mathbf{M}^+ = \mathbf{N}^+$ where **N** is a line in **Q.**■

The proof of the next result is entirely parallel to the argument for the previous corollary.

Corollary 6. *A subset* Π *of* $\mathbb{P}(\mathbb{R}^3)$ *is a plane if and only if there is a* $3-$ *dimensional vector subspace* **W** *of* $\mathbb{R}^4$ *such that a point* **X** *lies in* Π *if and only if its sets of homogeneous coordinates are nonzero vectors of* **W.**■

The next result is analogous to the two preceding corollaries, but it requires somewhat more work to prove. A reader who wishes to skip the proof may do so without loss of continuity or missing anything that will be needed later.

Theorem 7. *A subset* Λ *of* $\mathbb{P}(\mathbb{R}^3)$ *is a line if and only if there is a* $2-$ *dimensional vector subspace* **W** *of* $\mathbb{R}^4$ *such that a point* **X** *lies in* Λ *if and only if its sets of homogeneous coordinates are nonzero vectors of* **W.**

Before proving this result, we notice that it combines with a preceding corollary to yield the following uniform statement:

Summarizing statement. *If* $n = 2$ *or* 3, *then a subset* Λ *of* $\mathbb{P}(\mathbb{R}^n)$ *is a line if and only if there exists a* $2-$ *dimensional vector subspace* **W** *of* $\mathbb{R}^{n+1}$ *such that a*

point **X** *lies in* Λ *if and only if its sets of homogeneous coordinates are nonzero vectors of* **W.**■

Proof of Theorem 7. We shall first prove that for each line **L** there is a $2-$ dimensional vector subspace **W** with the desired properties. The proof splits into two cases, depending upon whether the line is an ordinary line or an ideal line.

Suppose first that **L** is the ordinary line containing the two distinct ordinary points **a** and **b** in $\mathbb{R}^3$, and consider the associated vectors $(\mathbf{a}, 1)$ and $(\mathbf{b}, 1)$ in $\mathbb{R}^4$. We claim these vectors are linearly independent; since they are both nonzero it suffices to check that neither is a multiple of the other. If they were, then one would have an equation of the form $(\mathbf{a}, 1) = c(\mathbf{b}, 1)$ where c is a nonzero scalar. Since $c(\mathbf{b}, 1) = (c\mathbf{b}, c)$, this implies $c = 1$, so that the two vectors must be equal and therefore $\mathbf{a} = \mathbf{b}$, contradicting our choices of these points.

Let **W** be the $2-$ dimensional subspace spanned by $(\mathbf{a}, 1)$ and $(\mathbf{b}, 1)$. Then **W** contains homogeneous coordinates for every ordinary point of **L** because each such point can be written in the form $\mathbf{x} = t\mathbf{a} + (1-t)\mathbf{b}$ for some scalar t, so that its homogeneous coordinates satisfy $(\mathbf{x}, 1) = t(\mathbf{a}, 1) + (1-t)(\mathbf{b}, 1)$. Also, **W** contains homogeneous coordinates the ideal point $\mathbf{L}^+$ because the latter has homogeneous coordinates

$$(\mathbf{a} - \mathbf{b}, 0) = (\mathbf{a}, 1) - (\mathbf{b}, 1).$$

We now need to show that **W** does not contain homogeneous coordinates for any other ordinary or ideal point. Suppose we are given an arbitrary linear combination of the form $\mathbf{w} = u(\mathbf{a}, 1) + v(\mathbf{b}, 1)$ where u and v are scalars such that not both are equal to zero. There are two cases depending upon whether $u + v$ is equal to zero or not equal to zero. If $u + v$ is nonzero then we know that $(u+v)^{-1}\mathbf{w}$ also represents the same point and that if we write $\mathbf{w} = (\mathbf{z}, u+v)$, then $(u+v)^{-1}\mathbf{w}$ satisfies the following equation:

$$\frac{1}{u+v}w = \left(\frac{1}{u+v}z, 1\right) = \frac{u}{u+v}(a, 1) + \frac{v}{u+v}(b, 1)$$

But this means that **w** is a set of homogeneous coordinates for the point $(u+v)^{-1}\mathbf{z}$ and the latter is a point on the line **L** joining **a** to **b.** Now suppose that $u + v$ is equal to zero, so that $v = -u$ and both are nonzero. In this case we have

$$\mathbf{w} = u(\mathbf{a}, 1) - u(\mathbf{b}, 1) = (u\mathbf{a} - u\mathbf{b}, 0)$$

which means that **w** represents the ideal point on the line **L.**

Suppose now that **L** is an ideal line in $\mathbb{P}(\mathbf{Q})$ for some ordinary plane **Q** in $\mathbb{R}^3$. As in the proof of the previous theorem, express **Q** as $\mathbf{z} + \mathbf{W}$ for some $2-$ dimensional vector subspace **W** of $\mathbb{R}^3$. The proof of that theorem then shows that the points of $\mathbb{L}_\infty(\mathbf{Q})$ are precisely those points whose homogeneous coordinates have the form

$(\mathbf{w}, 0)$ for some nonzero vector $\mathbf{w}$ in $\mathbf{W}$. If we convert this into a statement involving linear homogeneous equations, we see that $\mathbb{L}\infty(\mathbf{Q})$ consists of all points whose homogeneous coordinates satisfy $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$ defining the extended ordinary plane $\mathbb{P}(\mathbf{Q})$ and together with the equation $x_4 = 0$ which defines the ideal plane. In the first equation we know that at least one of the coordinates a_1, a_2, a_3 is nonzero, so these equations are linearly independent and hence the vector subspace of solutions for this system will be $2-$ dimensional. Since this solution set contains the $2-$ dimensional subspace $\mathbf{W}$, it must be equal to $\mathbf{W}$, and therefore the ideal line $\mathbb{P}(\mathbf{Q})$ is the set of points whose homogeneous coordinates are nonzero vectors in $\mathbf{W}$.

This completes the proof that every line is describable as indicated in the theorem. Conversely, if we are given a $2-$ dimensional subspace $\mathbf{W}$, we have to show that it corresponds to a line. There are two cases, depending upon whether $\mathbf{W}$ contains any vectors whose fourth coordinates are nonzero. If it does not, then every vector in $\mathbf{W}$ has the form $(\mathbf{u}, 0)$, where $\mathbf{u}$ lies in some $2-$ dimensional subspace $\mathbf{V}$ of $\mathbb{R}^3$, and by the immediately preceding paragraph we see that the set of points with homogeneous coordinates in $\mathbf{W}$ is just the ideal line $\mathbb{L}\infty(\mathbf{V})$ of the ordinary plane determined by $\mathbf{V}$.

Finally, suppose that the $2-$ dimensional subspace $\mathbf{W}$ contains at least one vector whose fourth coordinate is nonzero. If we multiply such a vector by the inverse of its fourth coordinate, we see that $\mathbf{W}$ contains a vector of the form $(\mathbf{v}, 1)$ where $\mathbf{v}$ is some vector in $\mathbb{R}^3$. Since $\mathbf{W}$ is $2-$ dimensional, it also contains a second vector of the form $(\mathbf{w}, t)$ that is not a scalar multiple of $(\mathbf{v}, 1)$; if t is nonzero, then $(t^{-1}\mathbf{w}, 1)$ will also be a vector in $\mathbf{W}$ which is not a scalar multiple of $(\mathbf{v}, 1)$, and hence by earlier parts of this argument we know that the points whose homogeneous coordinates lie in $\mathbf{W}$ are the points on the projective extension of the ordinary line joining $\mathbf{v}$ and $t^{-1}\mathbf{w}$. On the other hand, if $t = 0$ then we claim that $(\mathbf{v}, 1) + (\mathbf{w}, 0) = (\mathbf{v} + \mathbf{w}, 1)$ is not a scalar multiple of $(\mathbf{v}, 1)$; if we know this, then it will follow that the points whose homogeneous coordinates lie in $\mathbf{W}$ are merely the points on the projective extension of the ordinary line joining $\mathbf{v}$ and $\mathbf{v} + \mathbf{w}$. To prove the claim, observe that if we have $(\mathbf{v} + \mathbf{w}, 1) = c(\mathbf{v}, 1)$ for some scalar c then we must have $c = 1$ because the last coordinates are equal, and hence we must also have $\mathbf{v} = \mathbf{v} + \mathbf{w}$. However, since we know that the vector $(\mathbf{w}, 0)$ is not a scalar multiple of $(\mathbf{v}, 1)$, it follows that $(\mathbf{w}, 0)$ and hence $\mathbf{w}$ must be nonzero, which contradicts the conclusion $\mathbf{v} + \mathbf{w} = \mathbf{w}$ from the preceding sentence. It follows that $(\mathbf{v} + \mathbf{w}, 1)$ is not a scalar multiple of $(\mathbf{v}, 1)$, and as noted above this completes the proof.■

Application to collinear and coplanar points

Homogeneous coordinates yield very simple determinant tests for three points in $\mathbb{P}(\mathbb{R}^2)$ to be collinear and for four points in $\mathbb{P}(\mathbb{R}^3)$ to be coplanar.

Proposition 8. **(1)** *Suppose that* **X, Y** *and* **Z** *are distinct points in* $\mathbb{P}(\mathbb{R}^2)$ *with homogeneous coordinates given by* **a, b** *and* **c** *respectively, and let* **[a, b, c]** *denote the determinant of the* 3×3 *matrix whose columns are given by* **a, b** *and* **c.** *Then* **X, Y** *and* **Z** *are collinear if and only if* **[a, b, c]** $= 0$.

(2) *Suppose that* **X, Y, Z** *and* **W** *are distinct points in* $\mathbb{P}(\mathbb{R}^3)$ *with homogeneous coordinates given by* **a, b, c** *and* **d** *respectively, and let* **[a, b, c, d]** *denote the determinant of the* 4×4 *matrix whose columns are given by* **a, b, c** *and* **d.** *Then* **X, Y, Z** *and* **W** *are coplanar if and only if* **[a, b, c, d]** $= 0$.

Proof. For the first part, note that the given three points in $\mathbb{P}(\mathbb{R}^2)$ are collinear if and only if their homogeneous coordinates **a, b** and **c** span a $2-$dimensional subspace, and this is true if and only if the three vectors are linearly dependent. Since the latter is equivalent to the equation **[a, b, c]** $= 0$, it follows that the original points are collinear if and only if this equation holds.

The proof of the second part is similar. The given four points in $\mathbb{P}(\mathbb{R}^3)$ are coplanar if and only if their homogeneous coordinates **a, b, c** and **d** span a $3-$dimensional subspace, and this is true if and only if the four vectors are linearly dependent. Since the latter is equivalent to the equation **[a, b, c, d]** $= 0$, it follows that the original points are coplanar if and only if this equation holds.■

Important special cases. Of course, one can apply these directly to ordinary points with their usual rectangular coordinates because an ordinary point with rectangular coordinates $(x_1, \ldots, x_n)$ has homogeneous coordinantes given by $(x_1, \ldots, x_n, 1)$.■

Incidence properties of projective spaces

In Section **II.1** of the notes we listed several incidence axioms describing some of the most basic facts about lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$. Since we have used the same names for certain subsets of $\mathbb{P}(\mathbb{R}^2)$ and $\mathbb{P}(\mathbb{R}^3)$, it is natural to ask if these new concepts of lines and planes also satisfy these simple conditions. Our objective here is to show that they do, but they also have some properties that are markedly different from those of Euclidean lines and planes. With the material we have developed, all the proofs turn out to be fairly simple.

Proposition 9. (Verification of Property I – 1) *Let* $n = 2$ *or* 3. *Given any two distinct points in* $\mathbb{P}(\mathbb{R}^n)$, *there is exactly one line that contains them.*

Proof. Let **X** and **Y** be the distinct points, and suppose that their homogeneous coordinates are the nonzero vectors in the $1-$dimensional vector subspaces **V** and **W** in $\mathbb{R}^{n+1}$. Let **U** be the vector subspace given by the linear sum **V** + **W**; since **X** and **Y** are distinct points, it follows that no nonzero vector in **V** is a multiple of a

vector in **W** and vice versa, so that **U** contains a basis of two vectors (one from **V** and one from **W**), so that it is $2-$ dimensional. Let **L** be the line in $\mathbb{P}(\mathbb{R}^n)$ determined by **U.** Then by construction we know that **X** and **Y** both lie on **L.** Suppose now that **L′** is an arbitrary line containing **X** and **Y,** and let **U′** be its associated $2-$ dimensional subspace. If **v** and **w** are nonzero vectors in **V** and **W** respectively, it follows that **v** and **w** must also lie in **U′,** so that **U** is contained in **U′.** Since we know that **U′** is $2-$ dimensional it follows that **U** = **U′** and hence **L** = **L′.**■

Proposition 10. (Verification of Property $\mathbf{I-2}$) *Let* $n = 2$ *or* 3. *Then every line in* $\mathbb{P}(\mathbb{R}^n)$ *contains at least two points.*

Proof. Let **L** be a line in $\mathbb{P}(\mathbb{R}^n)$, and let **U** be the $2-$ dimensional vector subspace of $\mathbb{R}^{n+1}$ corresponding to **L.** If $\{\mathbf{v}, \mathbf{w}\}$ forms a basis for **U,** let **X** and **Y** be the points in $\mathbb{P}(\mathbb{R}^n)$ corresponding to **v** and **w** respectively. It follows that **X** and **Y** are distinct points on the line **L.**■

The same argument employed in Section $\mathbf{II.1}$ yields the following simple fact about lines in projective spaces.

Proposition 11. *Let* $n = 2$ *or* 3. *Then two distinct lines in* $\mathbb{P}(\mathbb{R}^n)$ *have at most one point in common.*■

We now proceed to the axioms which mainly involve $3-$ dimensional space(s).

Proposition 12. (Verification of Property $\mathbf{I-3}$) *Given three distinct points in* $\mathbb{P}(\mathbb{R}^3)$ *that are not contained in a line, there is exactly one plane in* $\mathbb{P}(\mathbb{R}^3)$ *that contains them.*

Proof. Let **X, Y** and **Z** be the three noncollinear points, and suppose that **u, v** and **w** are sets of homogeneous coordinates for **X, Y** and **Z** respectively.

The noncollinearity hypothesis implies that there is no $2-$ dimensional vector subspace containing all three vectors; if these vectors were linearly dependent, then there would be such a vector subspace, so it follows that **u, v** and **w** must be linearly independent. Hence they span a $3-$ dimensional subspace **S,** and if **P** is the plane corresponding to **S** then the three original points **X, Y** and **Z** must lie in **S.** Conversely, if **Q** is a plane containing **X, Y** and **Z** and **T** is the $3-$ dimensional vector subspace corresponding to **Q,** then it follows that **u, v** and **w** all lie in **T,** so that **T** must contain **S.** Since both are $3-$ dimensional, they must be equal and therefore we have **P** = **Q.**■

Proposition 13. (Verification of Property $\mathbf{I-4}$) *Let* $n = 2$ *or* 3. *Then every plane in* $\mathbb{P}(\mathbb{R}^n)$ *contains at least three distinct points.*

Proof. Let **P** be the plane, let **W** be the $3-$dimensional vector subspace of $\mathbb{R}^4$ corresponding to **P,** let $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ be a basis for **W,** and take the points **X, Y** and **Z** represented by **a, b** and **c** respectively. Then **X, Y** and **Z** are three distinct points of **P.**■

Proposition 14. (Verification of Property $\mathbf{I-5}$) *If* **L** *is a line in* $\mathbb{P}(\mathbb{R}^3)$ *and two distinct points* **A, B** *on* **L** *also lie on the plane* **P** *in* $\mathbb{P}(\mathbb{R}^3)$, *then all points of the line* **L** *are contained in* **P.**

Proof. Let **P** be the plane, let **W** be the $3-$dimensional vector subspace $\mathbb{R}^4$ corresponding to **P,** let **u** and **v** be homogeneous coordinates for **A** and **B** respectively; then **u** and **v** span a $2-$dimensional subspace **U** which must be contained in **W.** By the proof of the first incidence property we know that a point lies on **AB** if and only if its homogeneous coordinates lie in **U.** Therefore if **X** lies on **L,** then its homogeneous coordinates lie in **U** and hence also in **W,** so that **X** must also lie in the plane **P.**■

Proposition 15. (Verification of Property $\mathbf{I-6}$) *If two distinct planes in* $\mathbb{P}(\mathbb{R}^3)$ *have one point in common, then their intersection is a line.*

Proof. Let **P** and **Q** be the planes, and let **V** and **W** be the $3-$dimensional vector subspace $\mathbb{R}^4$ corresponding to **P** and **Q.** By elementary linear algebra we know that

$$\dim(\mathbf{V} \cap \mathbf{W}) \geq \dim \mathbf{V} + \dim \mathbf{W} - \dim \mathbb{R}^4 = 3 + 3 - 4 = 2$$

and since the intersection is contained in the $3-$dimensional subspaces **V** and **W** it follows that the dimension of $\mathbf{V} \cap \mathbf{W}$ is either 2 or 3; if the latter holds, then we have $\mathbf{V} = \mathbf{V} \cap \mathbf{W} = \mathbf{W},$ which means that **P** and **Q** are not distinct. Therefore the intersection is a $2-$dimensional vector subspace, so that the set of points represented by nonzero vectors in $\mathbf{V} \cap \mathbf{W}$ — which is just the set of points in the intersection $\mathbf{P} \cap \mathbf{Q}$ — must be a line.■

Notice that we have really shown something that is much stronger than property $\mathbf{I-6}$:

Theorem 16. *If* **P** *and* **Q** *are distinct planes in* $\mathbb{P}(\mathbb{R}^3)$, *then their intersection is a line.*■

Of course, the corresponding statement for planes in Euclidean $3-$dimensional space is false because there are many pairs of parallel planes. We should note that if **Q** and **Q′** are parallel ordinary planes in $\mathbb{R}^3$, then the intersection of the extended planes $\mathbb{P}(\mathbf{Q})$ and $\mathbb{P}(\mathbf{Q}')$ in $\mathbb{P}(\mathbb{R}^3)$ is their common line at infinity $\mathbb{L}(\mathbf{Q}) = \mathbb{L}(\mathbf{Q}')$. Likewise, if **Q** is an extended ordinary plane and $\mathbb{P}(\infty)$ is the ideal plane, then $\mathbb{P}(\mathbf{Q})$ and $\mathbb{P}(\infty)$ meet in the ideal line $\mathbb{L}_\infty(\mathbf{Q})$.■

In particular, there are no "parallel planes" in $\mathbb{P}(\mathbb{R}^3)$, and the next result shows that there are also no "parallel lines" in $\mathbb{P}(\mathbb{R}^2)$.

Theorem 17. *Let* $n = 2$ *or* 3, *and let* **L** *and* **M** *are distinct coplanar lines in* $\mathbb{P}(\mathbb{R}^n)$. *Then* **L** *and* **M** *have a point in common.*■

Note. **The assumption that the lines be coplanar cannot be dropped.** For example, if $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the standard unit vectors in $\mathbb{R}^3$, then the extensions of the two ordinary lines $\mathbf{0e}_1$ and $\mathbf{e}_2\mathbf{e}_3$ do not have any ordinary or ideal points in common (checking this is left to the exercises).

Proof. Let **P** be the plane containing **L** and **M**, let **W** be the $3-$dimensional vector subspace corresponding to **P**, and let **U** and **V** be the $2-$dimensional vector subspaces corresponding to **L** and **M** respectively. Then **U** and **V** are contained in **W** and hence their linear sum is too (in fact, one has $\mathbf{W} = \mathbf{U} + \mathbf{V}$ but we shall not need this). As before, by elementary linear algebra we know that

$$\dim(\mathbf{U} \cap \mathbf{V}) \geq \dim \mathbf{U} + \dim \mathbf{V} - \dim \mathbf{W} = 2 + 2 - 3 = 1$$

and therefore there is a nonzero vector **z** in $\mathbf{U} \cap \mathbf{V}$. If **X** is the point represented by **z**, it follows that **X** lies on both **L** and **M.**■

Of course, if we have two extensions of ordinary parallel lines, then these extended lines will meet at the ideal point associated to both of the original parallel lines. Furthermore, the ideal line of the plane meets an extended ordinary line at the latter's ideal point.■

One can now prove the following results using exactly the same arguments presented in Section **II.1.**

Proposition 18. *If* **L** *is a line and* **X** *is a point not on* **L**, *then there is a unique plane* **P** *which contains* **X** *and (all the points of)* **L.**■

Proposition 19. *If* **L** *and* **M** *are distinct lines that have one point in common, then there is a unique plane* **P** *which contains both* **L** *and* **M.**■

In other words, two intersecting lines ***in projective space*** determine a unique plane.

IV.4 : Projective duality

> By the principle of duality ... geometry at one stroke is doubled in extent with no expenditure of extra labor.
>
> E. T. Bell (1883 – 1960), ***Men of Mathematics***

Interior decorators often use a mirror covering an entire wall to make a room look larger. Duality principles in mathematics are conceptual mirrors which similarly broaden several parts of mathematics, taking every valid result in an area and yielding an equally valid mirror image (or **dual version**) of it. Projective geometry has important duality

properties that were formulated explicitly and used very effectively by Poncelet and Gergonne.

Our treatment of duality in projective geometry is aimed at giving a rapid introduction to the most basic aspects of the idea; more comprehensive (and generally more elegant) accounts are given in virtually every book on projective geometry.

An example of duality

The operation which sends a matrix to its transpose has a curious significance. Since the rows and columns of the transposed matrix $^{\mathrm{T}}\mathbf{A}$ are respectively the columns and rows of the original matrix $\mathbf{A}$, there are situations where one can use a fact about rows and columns to prove a fact about columns and rows. This depends heavily on some basic properties of the transposition operation:

> *Transposition of matrices satisfies the algebraic identities* $^{\mathrm{T}}(\mathbf{A}+\mathbf{B}) = {}^{\mathrm{T}}\mathbf{A} + {}^{\mathrm{T}}\mathbf{B}$, $^{\mathrm{T}}(c\mathbf{A}) = c({}^{\mathrm{T}}\mathbf{A})$, $^{\mathrm{T}}(\mathbf{AB}) = {}^{\mathrm{T}}\mathbf{B}\,{}^{\mathrm{T}}\mathbf{A}$, *and* $^{\mathrm{T}}\mathbf{I} = \mathbf{I}$ *If* $\mathbf{I}$ *is an identity matrix. Furthermore, the transpose of a transposed matrix is just the original matrix.*

Given a statement about matrix algebra, the ***dual statement*** about matrices is formed by **(1)** switching the roles of rows and columns for all the matrices which appear, and **(2)** reversing the orders of all matrix multiplications. We then have the following basic **metamathematical** (MET – a – mathematical) fact (the highlighted word means "about mathematics"):

Metatheorem. *A statement about matrix algebra is true if and only if its dual statement is true.*

Explanation. Given a statement about matrices, apply it to the transposed matrices. If we do so then rows become columns and vice versa, and the orders of multiplications are reversed. Since the original statement is true and every matrix is the transpose of some other matrix, the original statement applied the transposed matrices — which is precisely the dual statement — must be true as well. Similarly, if the dual statement is true, then its double dual is also true. But the double dual is just the original statement, so this means the original statement is true.■

Example. The algebraic statement that $\mathbf{B}$ and $\mathbf{C}$ are $m \times n$ and $n \times m$ matrices such that $\mathbf{CB} = \mathbf{I}$ dualizes to an algebraic statement that $\mathbf{B}$ and $\mathbf{C}$ are $n \times m$ and $m \times n$ matrices such that $\mathbf{BC} = \mathbf{I}$. Consider now the following statement:

> *If the rows of a matrix* $\mathbf{B}$ *are linearly independent, then there is a matrix* $\mathbf{C}$ *such that* $\mathbf{BC} = \mathbf{I}$.

If we apply the previous rules, we obtain the dual statement:

> *If the columns of a matrix* $\mathbf{B}$ *are linearly independent, then there is a matrix* $\mathbf{C}$ *such that* $\mathbf{CB} = \mathbf{I}$.

Therefore, if we can prove one of these statements, we automatically know the other is true. In this particular example it is not difficult to prove each statement separately without appealing to the duality principle, but it is at least conceivable that there will be

more difficult theorems for which dualization is at least a labor saving device and at best yields new conclusions that might not have been otherwise easy to anticipate.

Duality in projective spaces

Before proceeding to the main topic of this section, we mention a default convention that has been at least implicit in these notes:

For many computational purposes, we are viewing vectors in $\mathbb{R}^n$ as $n \times 1$ matrices (also known as column vectors).

In particular, this is useful if we are computing with linear transformations, for it allows us to view a linear transformation **L** from $\mathbb{R}^n$ to $\mathbb{R}^m$ as being given an $m \times n$ matrix **A** such that for every vector **v** in $\mathbb{R}^n$ we have **L(v) = Av.** For the rest of this unit we shall use this convention very explicitly.

Homogeneous coordinates for lines and planes. We have seen that lines in $\mathbb{P}(\mathbb{R}^2)$ and planes in $\mathbb{P}(\mathbb{R}^3)$ are describable as sets of points whose homogeneous coordinates satisfy a nontrivial linear homogeneous equation in three or four variables. The following result formalizes this fact.

Theorem 1. *Let $n = 2$ or 3, and suppose that* **H** *is a line in $\mathbb{P}(\mathbb{R}^2)$ if $n = 2$ or that* **H** *is a plane in $\mathbb{P}(\mathbb{R}^3)$ if $n = 3$. Then there is a nonzero $1 \times n$ row vector* **a** *such that a point* **Y** *belongs to* **H** *if and only if a set* **x** *of homogeneous coordinates for* **Y** *satisfies the matrix algebra equation* **ax = 0** (*where we identify 1×1 matrices with their unique scalar entries*). *Furthermore, two nonzero row vectors define the same subset* **H** *if and only one is a nonzero multiple of the other.*

As usual, we note that if one set **x** of homogeneous coordinates for **Y** satisfies the matrix algebra equation **ax = 0,** then the same equation is satisfied by every set of homogeneous coordinates for **Y.**

In the setting above, we shall say that any nonzero row vector which satisfies the condition in the theorem is a set of ***homogeneous coordinates*** for **H.**

Proof. We know that **H** is definable as the set of points whose homogeneous coordinates satisfy a nontrivial linear homogeneous equation of the form

$$a_1 x_1 + \ldots + a_{n+1} x_{n+1} = 0$$

in which at least one of the coefficients a_k is nonzero. We can take the nonzero $1 \times n$ row vector **a** so that its entries are given by these coefficients. If c is a nonzero constant then clearly the equation $ca_1 x_1 + \ldots + ca_{n+1} x_{n+1} = 0$ is equivalent to the equation displayed above, so if **b** $= c$**a** for some nonzero scalar c then both **a** and **b** define the same subset **H.** Conversely, suppose that **a** and **b** define the same subset **H,** and let **W** be the n – dimensional vector subspace of $\mathbb{R}^{n+1}$ which

corresponds to **H.** Then the only way that **W** can be the set of solutions for both equations $\mathbf{ax} = \mathbf{0}$ and $\mathbf{bx} = \mathbf{0}$ is if **a** and **b** are nonzero multiples of each other, and therefore the row vector **a** is uniquely defined up to multiplication by a nonzero scalar.■

Example. Suppose that **X** and **Y** are distinct points in $\mathbb{P}(\mathbb{R}^2)$ and that the vectors **a** and **b** in $\mathbb{R}^3$ are homogeneous coordinates for these points. If **L** is the line containing **X** and **Y,** then ***homogeneous coordinates for*** **L** ***are given by the transpose of the cross product*** $^{\mathbf{T}}(\mathbf{a} \times \mathbf{b})$; this is true because $\mathbf{X} \neq \mathbf{Y}$ implies **a** and **b** are linearly independent, so that the cross product is nonzero and perpendicular to both **a** and **b.**

The preceding theorem has an interesting consequence which is fundamental to projective geometry.

Theorem 2. (*Coordinate Duality Principle*) *Let* $n = 2$ *or* 3, *and define* $\mathbb{P}(\mathbb{R}^n)^{\#}$ *to be the set of all lines in* $\mathbb{P}(\mathbb{R}^n)$ *if* $n = 2$ *and the set of all planes in* $\mathbb{P}(\mathbb{R}^n)$ *if* $n = 3$ *(we shall generically call the lines or planes* ***hyperplanes****). Then there is a* $1-1$ *correspondence* π *from* $\mathbb{P}(\mathbb{R}^n)$ *to* $\mathbb{P}(\mathbb{R}^n)^{\#}$ *such that a point* **X** *in* $\mathbb{P}(\mathbb{R}^n)$ *lies on a hyperplane* **H** *in* $\mathbb{P}(\mathbb{R}^n)^{\#}$ *if and only if the point* $\pi^{-1}[\mathbf{H}]$ *in* $\mathbb{P}(\mathbb{R}^n)$ *lies on the hyperplane* $\pi[\mathbf{X}]$ *in* $\mathbb{P}(\mathbb{R}^n)^{\#}$.

Proof. Given a point **X,** let **v** be a set of homogeneous coordinates for **X** and take **H** to be the line or plane whose coordinates are the transpose of **v.** Although this involves choosing a set of homogeneous coordinates, the line or plane **H** does not depend upon the choice, for every other set of homogeneous coordinates for **X** is given by $c\mathbf{a}$ for some nonzero scalar c and the transposes of **a** and $c\mathbf{a}$ define the same object **H** in $\mathbb{P}(\mathbb{R}^n)^{\#}$. Since transposition determines a $1-1$ correspondence from column vectors to row vectors which preserves all vector space operations, it follows that the map π defines a $1-1$ correspondence from $\mathbb{P}(\mathbb{R}^n)$ to $\mathbb{P}(\mathbb{R}^n)^{\#}$.

We shall now verify the extra condition on π stated in the theorem. Given $\mathbf{X} \in \mathbb{P}(\mathbb{R}^n)$ and $\mathbf{H} \in \mathbb{P}(\mathbb{R}^n)^{\#}$, let **b** and **a** denote homogeneous coordinates for **X** and **H** respectively, so that $\mathbf{X} \in \mathbf{H}$ if and only if $\mathbf{ba} = \mathbf{0}$. By construction we know that $\pi[\mathbf{X}]$ and $\pi^{-1}[\mathbf{H}]$ have homogeneous coordinates given by $^{\mathbf{T}}\mathbf{b}$ and $^{\mathbf{T}}\mathbf{a}$ respectively, and therefore $\mathbf{X} \in \mathbf{H}$ implies

$$^{\mathbf{T}}\mathbf{a}\,^{\mathbf{T}}\mathbf{b} = {}^{\mathbf{T}}(\mathbf{ba}) = {}^{\mathbf{T}}\mathbf{0} = \mathbf{0}$$

so that $\pi^{-1}[\mathbf{H}] \in \pi[\mathbf{X}]$. Conversely, if the latter holds then $\mathbf{0} = {}^{\mathbf{T}}\mathbf{a}\,^{\mathbf{T}}\mathbf{b} = {}^{\mathbf{T}}(\mathbf{ba})$ implies $\mathbf{ba} = \mathbf{0}$, so that $\mathbf{X} \in \mathbf{H}$.■

In effect, the mapping π sends a point in $\mathbb{P}(\mathbb{R}^n)$ to a line or plane, and the set of all points on a line **H** $(n = 2)$ or plane **H** $(n = 3)$ is sent to the set of all lines or planes which pass through the point π^{-1}**[H].** It is natural to ask what happens to the set of all points on a line **L** in $\mathbb{P}(\mathbb{R}^3)$ under the mapping π**.** In order to do this, we need some input from linear algebra.

Lemma 3. *Let* **W** *be a* $k-$*dimensional vector subspace of* $\mathbb{R}^m$*, and let* **V** *be the set of all vectors that are perpendicular to every vector in* **W.** *Then* **V** *is an* $(m-k)-$*dimensional vector subspace of* $\mathbb{R}^m$**.** *Furthermore,* **W** *is the set of all vectors that are perpendicular to every vector in* **V.**

Proof. Let $\mathbf{w}_1, \ldots, \mathbf{w}_k$ be a basis for **W.** Then a vector **x** belongs to **V** if and only if **x** is perpendicular to each of the vectors $\mathbf{w}_j$**.** To see this, observe that if **x** is perpendicular to every vector in **W** then it will be perpendicular to every vector in the basis, and conversely if $\mathbf{w}_j \cdot \mathbf{x} = 0$ for all j, then we can express $\mathbf{w} \in \mathbf{W}$ as a linear combination $\mathbf{w} = c_1\mathbf{w}_1 + \ldots + c_k\mathbf{w}_k$ for suitable scalars $\mathbf{w}_j$, and hence we have

$$\mathbf{w} \cdot \mathbf{x} = (c_1\mathbf{w}_1 + \ldots + c_k\mathbf{w}_k) \cdot \mathbf{x} = c_1(\mathbf{w}_1 \cdot \mathbf{x}) + \ldots + c_k(\mathbf{w}_k \cdot \mathbf{x}) = 0$$

so that $\mathbf{x} \in \mathbf{V}$**.** It follows that **V** is the solution set for a system of k independent linear homogeneous equations, and therefore **V** is a vector subspace whose dimension is equal to $(m-k)$**.**

To prove the statement in the final sentence of the theorem, let **U** be the set of vectors that are perpendicular to every vector in **V;** then the preceding discussion shows that **U** is a vector subspace whose dimension is equal to k**.** However, we also know that **U** contains **W** and **W** is also a vector subspace whose dimension is equal to k, and thus we have **U = W.**■

We are now ready to state and prove a result on what happens to lines in $\mathbb{P}(\mathbb{R}^3)$ under the mapping π**.** This proof is more delicate than the others in this section, and for most purposes it is enough to understand the statement of the result (including the alternate version following the proof).

Proposition 4. *Let* $n = 3$*, and let* **L** *be a line in* $\mathbb{P}(\mathbb{R}^3)$**.** *Then the set of all planes expressible as* π**[X],** *where* **X** *ranges over all the points of* **L,** *is sent to the set of all planes which contain some (possibly different) line* **M.** *Conversely, if* **M** *is a line in* $\mathbb{P}(\mathbb{R}^3)$ *and* $\mathcal{E}_{\mathbf{M}}$ *is the set of all planes in* $\mathbb{P}(\mathbb{R}^3)$ *which contain the line* **M,** *then the image of* $\mathcal{E}_{\mathbf{M}}$ *under* π^{-1} *is the set of all points on some (possibly different) line* **L.**

Proof. If **L** is a line, then we know there is a $2-$dimensional vector subspace **W** of $\mathbb{R}^4$ such that **X** lies in **L** if and only if its homogeneous coordinates lie in **W.** Let

V be the vector subspace of all $\mathbf{x} \in \mathbb{R}^4$ such that **x** is perpendicular to every vector in **W**; by the lemma we know that **V** is also $2-$ dimensional.

Let **M** be the line in $\mathbb{P}(\mathbb{R}^3)$ which corresponds to **V**, suppose that **X** is a point on **L**, and let **a** be a set of homogeneous coordinates for **X**; since $\mathbf{X} \in \mathbf{L}$, it follows that $\mathbf{a} \in \mathbf{W}$. The conditions on **W** and **V** imply that $\mathbf{0} = \mathbf{a} \cdot \mathbf{v} = {}^{\mathrm{T}}\mathbf{a}\,\mathbf{v}$ for all $\mathbf{v} \in \mathbf{V}$; therefore if **Y** is a point in **M** (so that its homogeneous coordinates lie in **V**), we know that **Y** lies on the plane $\pi[\mathbf{X}]$, and hence **M** is contained in $\pi[\mathbf{X}]$. Conversely, suppose that we have a plane **H** which contains **M**; let **u** be a set of homogeneous coordinates for the plane **H**. Then $\mathbf{u}\,\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in \mathbf{V}$, and by the Lemma we know that ${}^{\mathrm{T}}\mathbf{u} \in \mathbf{W}$. If **X** is the point represented by ${}^{\mathrm{T}}\mathbf{u}$, then it follows that $\mathbf{X} \in \mathbf{L}$ so that $\mathbf{H} = \pi[\mathbf{X}]$, where $\mathbf{X} \in \mathbf{L}$. This proves the first part of the theorem.

We now consider the second assertion in the theorem. By the argument at the end of the preceding paragraph, we know that π^{-1} sends the set of all planes **H** containing a line **M** into the set of all points **X** contained in a possibly different line **L**. Similarly, the argument at the beginning of the preceding paragraph shows that if $\mathbf{X} = \pi^{-1}[\mathbf{H}]$ is a point of **L**, then **H** contains the line **M**. This completes the proof of the second part of the theorem.■

Another way of stating the conclusion of the proposition is that π and π^{-1} map sets of points on a given line to sets of planes containing a (possibly different) given line and vice versa.

The preceding results lead to the following general Duality Principles.

Metatheorem 5. (Duality principle for the projective plane) *Every theorem about points and lines in* $\mathbb{P}(\mathbb{R}^2)$ *remains true if one interchanges the words* "*point*" *and* "*line*" *as well as the phrases* "*point lies on a line*" *and* "*line contains a point*."

Explanation. Many of the results of this section on $\mathbb{P}(\mathbb{R}^2)$ can be summarized in the table on the next. In each row of this table we have the following:

- **(a)** the geometric concept in the first column translates into the algebraic data in the second by interpreting everything in terms of coordinates,
- **(b)** the data in the second column translate into the data in the third column by switching the roles of row and column vectors,
- **(c)** the algebraic data in the third column correspond to the geometric data in the fourth column by translating algebraic language into geometric language by means or coordinates.

Geometric Concept	Algebraic Counterpart	Transposed Version	Dual Geometric Interpretation
Line	$\mathbf{1}$ – dimensional subspace of row vectors	$\mathbf{1}$ – dimensional subspace of column vectors	Point
Point	$\mathbf{1}$ – dimensional subspace of column vectors	$\mathbf{1}$ – dimensional subspace of row vectors	Line
A point lies on a line	The product of a column vector left multiplied by a row vector is zero	The product of a row vector left multiplied by a column vector is zero	A line contains a point
A line contains a point	The product of a row vector left multiplied by a column vector is zero	The product of a column vector left multiplied by a row vector is zero	A point lies on a line

A geometric statement involving concepts in the first column will be true if and only if its coordinate translation into the algebraic terms of the second column is true. Likewise, the properties of matrix transposition show that a statement involving concepts in the second column will be true if and only if the statement involving the matching concepts in the third column is true. Finally, an algebraic statement involving concepts in the third column will be true if and only if its coordinate translation into the geometric terms of the fourth column is true. Combining these, we see that a statement involving the geometric concepts in the first column will be true if and only if the statement involving the matching concepts in the fourth column is true.■

The corresponding principle in three dimensions is similar but involves more data.

Metatheorem 6. (Duality principle for projective $3-$ space) *Every theorem about lines and planes in* $\mathbb{P}(\mathbb{R}^3)$ *remains true if one interchanges the words "point" and "plane" as well as the following phrase pairs*:

(1) "*point lies on a plane*" *and* "*line contains a point*,"

(2) "*point lies on a line*" *and* "*plane contains a line*."

Explanation. We shall use the same general pattern of argument as before. In this case many of the results of this section on $\mathbb{P}(\mathbb{R}^3)$ can be summarized in the table on the next page:

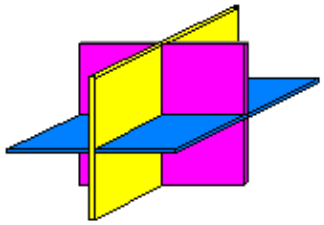
Geometric Concept	Algebraic Counterpart	Transposed Version	Dual Geometric Interpretation
Plane	$\mathbf{1}$ – dimensional subspace of row vectors	$\mathbf{1}$ – dimensional subspace of column vectors	Point
Point	$\mathbf{1}$ – dimensional subspace of column vectors	$\mathbf{1}$ – dimensional subspace of row vectors	Plane
A point lies on a plane	The product of a column vector left multiplied by a row vector is zero	The product of a row vector left multiplied by a column vector is zero	A plane contains a point
A plane contains a point	The product of a row vector left multiplied by a column vector is zero	The product of a column vector left multiplied by a row vector is zero	A point lies on a plane
Line (given by the set of points lying on it)	$\mathbf{2}$ – dimensional subspace of column vectors	$\mathbf{2}$ – dimensional subspace of row vectors	Line (given by the bundle of planes containing it)
A point lies on a line	A given column vector lies in a $\mathbf{2}$ – dimensional subspace of column vectors	A given row vector lies in a $\mathbf{2}$ – dimensional subspace of row vectors	A plane contains a line
A line contains a point	A $\mathbf{2}$ – dimensional subspace of column vectors contains a given column vector	A $\mathbf{2}$ – dimensional subspace of row vectors contains a given row vector	A line lies on a plane
A line lies on a plane	The right side product of a given row vector with every member of $\mathbf{2}$ – dimensional subspace of column vectors is zero	The left side product of a given column vector with every member of $\mathbf{2}$ – dimensional subspace of row vectors is zero	A line contains a point
A plane contains a line	A given row vector lies in a $\mathbf{2}$ – dimensional subspace of row vectors	A given column vector lies in a $\mathbf{2}$ – dimensional subspace of column vectors	A point lies on a line

The same reasoning as in the previous metatheorem shows that a statement involving the geometric concepts in the first column will be true if and only if the statement involving the matching concepts in the fourth column is true.■

Examples of dualization

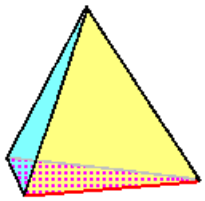
Here is a simple but basic example in which geometrical data are dualized. Consider the statement tnat **A, B, C** are noncollinear points. In $\mathbb{P}(\mathbb{R}^2)$ the dual configuration

is given by three lines **L, M, N** that do not pass through a common point; in the other words, the lines **L, M, N** are not concurrent. Similarly, in $\mathbb{P}(\mathbb{R}^3)$ the dual configuration is given by three planes **P, Q, S** that do not contain a common line. The drawing below depicts a triple of planes with this property.



(Modified from http://home.flash.net/~fmco/images/image82.gif)

Similarly, the dual to a configuration of four noncoplanar points **A, B, C, D** in $\mathbb{P}(\mathbb{R}^3)$ will be four planes **P, Q, S, T** whose intersection is empty. The simplest model for this is given by the planes containing the four faces of a triangular pyramid (or ***tetrahedron***).



We shall conclude this section with illustrations of how the duality principles are used. The first examples only involve homogeneous coordinates, but they are extremely useful for computational purposes (*even in ordinary coordinate geometry*!).

Proposition 7. **(1)** *Suppose that* **L, M** *and* **N** *are distinct lines in* $\mathbb{P}(\mathbb{R}^2)$ *with homogeneous coordinates given by* **a, b** *and* **c** *respectively, and let* **[a, b, c]** *denote the determinant of the* 3×3 *matrix whose rows are given by* **a, b** *and* **c.** *Then the lines* **L, M** *and* **N** *are concurrent if and only if* **[a, b, c]** $= 0$.

(2) *Suppose that* **S, T, U** *and* **V** *are distinct planes in* $\mathbb{P}(\mathbb{R}^3)$ *with homogeneous coordinates given by* **a, b, c** *and* **d** *respectively, and let* **[a, b, c, d]** *denote the determinant of the* 4×4 *matrix whose rows are given by* **a, b, c** *and* **d.** *Then the planes* **S, T, U** *and* **V** *contain a common point if and only if* **[a, b, c, d]** $= 0$.

Clearly this result is analogous to a proposition in the preceding section, and in fact it gives the dual statements to the earlier results. As in the preceding section, ***there are immediate applications to ordinary lines in*** $\mathbb{R}^2$ ***and planes in*** $\mathbb{R}^3$, for if the lines or planes are all described in terms of nontrivial linear equations, then homogeneous coordinates for the line or plane are given directly by the constant term together with the coefficients of the $x-$, $y-$, and (if applicable) $z-$ coordinates.

Proof. In the first case, if there is a point **Y** which lies on all three lines and it has homogeneous coordinates given by **z,** then we know that **a, b** and **c** all satisfy the

matrix equation $\mathbf{xz} = \mathbf{0}$. Since the subspace of all solutions to this equation is $2-$ dimensional, it follows that the three vectors must be linearly dependent and hence we have $\mathbf{[a, b, c]} = \mathbf{0}$. Conversely, if this equation holds then the three vectors span a $2-$ dimensional subspace and hence there is a nonzero column vector $\mathbf{z}$ such that

$$\mathbf{az} = \mathbf{bz} = \mathbf{cz} = \mathbf{0}.$$

If $\mathbf{Y}$ is a point whose homogeneous coordinates are given by $\mathbf{z}$, then it follows that $\mathbf{Y}$ lies on each of the lines $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

The proof in the $3-$ dimensional case is similar, the main differences being that one must make the following subsitutions:

planes **in place of** lines

four **in place of** three

$\mathbf{[a, b, c, d]} = \mathbf{0}$ **in place of** $\mathbf{[a, b, c]} = \mathbf{0}$

$\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ **in place of** $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$

$3-$ dimensional **in place of** $2-$ dimensional

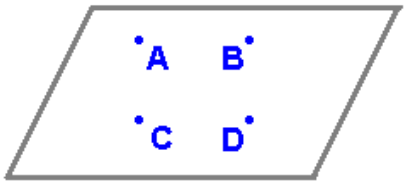
The details of carrying this out are left to the reader.■

Finally, here is an example involving the proof of a theorem in projective geometry and the formulation of its dual statements. The result itself is fairly simple, but the point is to show how dualization works in practice with a minimum of other complications.

We are interested in the following property of planes in projective geometry.

Proposition 8. *Let* $n = 2$ *or* 3, *and let* $\mathbf{P}$ *be a plane in* $\mathbb{P}(\mathbb{R}^n)$. *Then there exist four points in* $\mathbf{P}$ *such that no three are collinear.*

One particularly simple example of such a set of four points is depicted below:



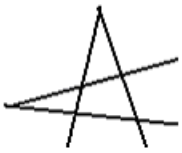
Proof. Suppose that the plane $\mathbf{P}$ corresponds to the $3-$ dimensional subspace $\mathbf{S}$ of $\mathbb{R}^{n+1}$, let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be a basis for $\mathbf{S}$, and suppose that these basis vectors represent the noncollinear points $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ respectively. Take $\mathbf{D}$ to be the point with homogeneous coordinates given by $\mathbf{u} + \mathbf{v} + \mathbf{w}$. We claim that no three of the points $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are collinear; this is true because simple linear algebra calculations show that each of the sets of vectors $\{\mathbf{u} + \mathbf{v} + \mathbf{w}, \mathbf{v}, \mathbf{w}\}$, $\{\mathbf{u}, \mathbf{u} + \mathbf{v} + \mathbf{w}, \mathbf{w}\}$, and $\{\mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w}\}$ is linearly independent.■

Here is what happens when we dualize the preceding result in the projective plane and in projective $3-$ space.

Plane Dual Proposition 9. *There exist four lines in* $\mathbb{P}(\mathbb{R}^2)$ *such that no three pass through a single point.*■

Space Dual Proposition 10. *Given an arbitrary point* **X** *in* $\mathbb{P}(\mathbb{R}^3)$, *there exist four planes containing* **X** *such that no three have a line in common.*■

The configurations in the dual propositions are illustrated below.



Dual Prop. 9

Dual Prop. 10

Note in particular that the $3-$dimensional dual configuration corresponds to the lateral sides of a pyramid with a rectangular base.