# V : Introduction to non – Euclidean geometry

> Over the course of the nineteenth century, under pressure of developments within mathematics itself, the accepted answer [ to questions like, "What is geometry**?**"] dramatically broke down. **…** Not since the ancient Greeks, if then, had there been such an irruption [or incursion] of philosophical ideas into the very heart of mathematics. **…** Mathematicians of the first rank **…** found themselves obliged to confront questions about **…** the status of geometry **…** The answers they gave did much to shape the mathematics of the twentieth century.
>
> W. Ewald (1954 –), ***Bulletin (New Series) of the American Mathematical Society***, Vol. **40** (2002), pp. 125 – 126.

We have already mentioned in Section **II.5** that the final assumption in Euclid's ***Elements*** (the so – called ***Fifth Postulate***) is far more complicated than the others. Furthermore, the proofs of the first **28** results in the ***Elements*** do not use the Fifth Postulate. In addition, there are general questions whether this postulate corresponds to physical reality because it involves objects which are too distant to be observed or questions about measurements that cannot necessarily be answered conclusively because there are always limits to the precision of physical measurements.

For these and other reasons, it is natural to speculate about the extent to which the exceptional Fifth Postulate is necessary or desirable as an assumption in classical geometry. Historical evidence suggests that such questions had been raised and debated extensively before Euclid's time, and for centuries numerous mathematicians tried to prove the Fifth Postulate from the others, or at least to find a simpler and more strongly intuitive postulate to replace it. During the 18$^{th}$ century several mathematicians made sustained efforts to resolve such issues by seeing what would happen if the Fifth Postulate were false, and in early 19$^{th}$ century a few mathematicians concluded that such efforts would not succeed and that there was a logically sound alternative to the truth of the Fifth Postulate. Later in that century other mathematicians proved results vindicating this conclusion; in particular, such results prove the logical impossibility of proving the Fifth Postulate or replacing it by something that raises fewer questions.

The discovery of non – Euclidean geometry had major implications for the role of geometry in mathematics, the sciences and even philosophy. The following three quotations summarize this change as it evolved from late in the 18$^{th}$ century through the beginning of the 20$^{th}$ century.

> The concept of [Euclidean] space is by no means of empirical origin, but is an inevitable necessity of thought.
>
> Kant, ***Critique of Pure Reason*** (1781).

I am convinced more and more that the necessary truth of our geometry cannot be demonstrated, at least not ***by*** the ***human*** intellect ***to*** the human understanding. Perhaps in another world, we may gain other insights into the nature of space which at present are unattainable to us. Until then we must consider geometry as of equal rank not with arithmetic, which is purely ***a priori***, but with mechanics.

Gauss, ***Letter to H. W. M. Olbers*** (1817). **[** ***Note:*** Olbers (1758 – 1840) was an astronomer, physician and physicist, and among other things he is known as the discoverer of the asteroid Pallas. **]**

One geometry cannot be more valid than another; it can only be more convenient.

Henri Poincaré (1854 – 1912), ***Science and Hypothesis*** (1901).

In this unit we shall discuss the mathematical theory (in fact, multiple theories) obtained by not assuming the Fifth Postulate, and we shall also include further comments on the role of geometry in modern mathematics and science.

# V.1 : Facts from spherical geometry

The sphere's perfect form has fascinated the minds of men for millennia. From planets to raindrops, nature … [makes use of] the sphere.

**http://www.bbc.co.uk/radio4/science/fiveshapes.shtml**

Spherical geometry can be said to be the first non – Euclidean geometry.

D. W. Henderson (1939 – ) & D. Taimina (1954 – ), ***Math. Assoc. of America Notes*** No. **68** (2005), p. 59.

Before we discuss the material generally known as non – Euclidean geometry, it will be helpful to summarize a few basic results from spherical geometry.

As noted in the following quote from **http://www.physics.csbsju.edu/astro/CS/CSintro.html**, it is natural to think of the sky as a large spherical dome and to use this as a basis for describing the positions of the stars and other heavenly bodies:

If you go out in an open field on a clear night and look at the sky, you have no indication of the distance to the objects you see. A particular bright dot may be an airplane a few miles off, a satellite a few hundred miles off, a planet a many millions of miles away, or a star more than a million times further away than the most distant planet. Since you can only tell direction (and not distance) you can imagine that the stars that you see are attached to the inside of a spherical shell that surrounds the Earth. The ancient Greeks actually believed such a shell really existed, but for us it is just a convenient way of talking about the sky.

In fact, the historical relationship between astronomy and spherical geometry goes much further than simple observations. When the first attempts at scientific theories of astronomy were developed by Eudoxus of Cnidus (408 – 355 B.C.E.) and Aristotle (384 – 322 B.C.E.) during the 4th century B.C.E., the ties between spherical geometry and astronomy became even closer, and both of these subjects were studied at length by later Greek scientists and mathematicians, including Hipparchus of Rhodes (190 – 120 B.C.E.), Heron of Alexandria (*c.* 10 A.D. – 75), Menelaus of Alexandria (70 – 130), and (last but not least) Claudius Ptolemy (85 – 165). Further information about ancient (and also modern) astronomy can be found at the following site:

http://phyun5.ucr.edu/~wudka/Physics7/Notes_www/web_notes.html

Another subject which began to emerge at the same time was trigonometry, which was studied both on the plane and on the surface of a sphere; not surprisingly, spherical trigonometry played a major role in efforts by ancient astronomers to explain the motions of stars and planets. In particular, during the Middle Ages both Arab and Indian mathematicians advanced spherical trigonometry far beyond the work of the ancient Greek mathematicians. During the later Middle Ages, practical questions about navigation began to influence the development of spherical geometry and trigonometry, and there was renewed interest which led to major advances in the subject continuing through the 18th century. Due to their importance for navigation and astronomy, spherical geometry and trigonometry were basic topics in high school mathematics curricula until the middle of the 20th century, but this has changed for several reasons (for example, one can use satellites and computers to do the work that previously required human vision and computation, and to do so more reliably). Our main purpose here is to describe the main aspects of spherical geometry, so some proofs and definitions will only be informal and other arguments will not be given at all.

## *Great circles*

In Euclidean plane and solid geometry, one reason for the importance of lines is that they describe the shortest paths between two points. On the surface of a sphere, the shortest curves between two points are given by ***great circles***.

<u>**Definitions.**</u> Given a point $\mathbf{X}$ in $\mathbb{R}^3$ and $k > 0$, the ***sphere*** $\Sigma$ ***of radius*** $k$ and center $\mathbf{X}$ is the set of all points $\mathbf{Y}$ in $\mathbb{R}^3$ such that $d(\mathbf{X}, \mathbf{Y}) = k$. A ***great circle*** on this sphere $\Sigma$ is a circle given by the intersection of $\Sigma$ with a plane containing $\mathbf{X}$.

The following result yields a large number of geometrically significant great circles.

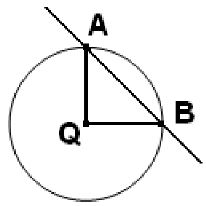
<u>**Proposition 1.**</u> *Let* $\Sigma$ *be a sphere as above with center* $\mathbf{X}$, *and let* $\mathbf{Y}$ *and* $\mathbf{Z}$ *be two points of* $\Sigma$.

1. *If* $\mathbf{X}$, $\mathbf{Y}$ *and* $\mathbf{Z}$ *are not collinear, then there is a unique great circle on* $\Sigma$ *containing* $\mathbf{Y}$ *and* $\mathbf{Z}$.
2. *If* $\mathbf{X}$, $\mathbf{Y}$ *and* $\mathbf{Z}$ *are collinear, then there are infinitely many great circles on* $\Sigma$ *containing* $\mathbf{Y}$ *and* $\mathbf{Z}$.

The second possibility arises when the segment **[YZ]** is a diameter of the sphere; in this case we often say that $\mathbf{Y}$ and $\mathbf{Z}$ are ***<u>antipodal</u>*** (pronounced ann–<u>TIP</u>–o–dal).

***<u>Sketch of proof.</u>*** The results follow by considering set of all planes containing the three points. In the first case there is only one, but in the second there are infinitely many planes containing the line **YZ.** ■

***Major and minor arcs.*** If **A** and **B** are points on a circle $\Gamma$, then they determine two arcs. If the points are antipodal, the two arcs are semicircles, and if they are not antipodal one has a ***major arc*** and a ***minor arc***. One way of distinguishing between these arcs is that the minor arc consists of **A, B** and all points on the circle $\Gamma$ which lie on the ***opposite side*** of **AB** as the center **Q**, while the major arc consists of **A, B** and all points on the circle $\Gamma$ which lie on the ***same side*** of **AB** as **Q**. We shall denote the minor arc by the symbol **€(AB)** (*Euro sign*).



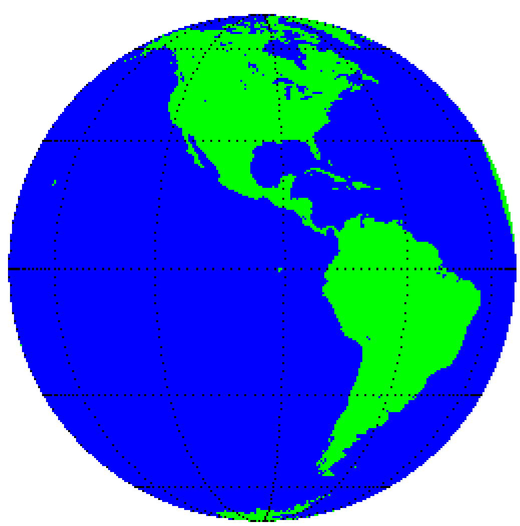

With this terminology we can describe the shortest curve(s) joining two points **A** and **B** on the sphere more precisely as follows: If the points are antipodal, the shortest curve is any semicircular arc joining **A** and **B** (such an arc is automatically a great circle), and if the points are not antipodal, it is the minor arc on the great circle determined by the points **A** and **B**. Actually proving these statements turns out to be a nonelementary exercise. One approach involves the theory of polyhedral angles from classical solid geometry and is summarized in the following note:

**http://math.ucr.edu/~res/math133/polyangles.pdf**

A different approach to the proof using differential calculus appears in the following online document:

**http://math.ucr.edu/~res/math138A/greatcircles.pdf**

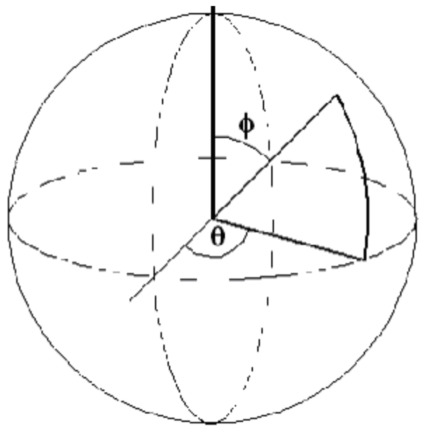
***Latitude, longitude and spherical coordinates.*** The standard method for locating points on the surface of the earth is by means of latitude and longitude coordinates.



In fact, these are equivalent to the spherical coordinates that are used in multivariable calculus. Specifically, we specialize spherical coordinates to the sphere of radius $a$, the conversion from rectangular to spherical coordinates is given by

$$(x, y, z) \quad = \quad (a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ a\cos\phi)$$

Then $\theta$ corresponds to the longitude counterclockwise from the meridian semicircle through $(0, 0, 1)$, $(0, 1, 0)$ and $(0, 0, -1)$, and $\phi$ corresponds to rescaled latitude, where $0$ degrees represents the north pole and $180$ degrees represents the south pole.




(***Source:*** **http://www.math.uncc.edu/~droyster/courses/fall96/math3181/notes/hyprgeom.html**)

*Lunes*

Straight lines provide the basic pieces with which constructs familiar plane figures, and similarly great circle arcs provide the basic pieces for constructing spherical figures. Some of these are analogous to figures in the plane, but we shall start with one class of figures that is different. Portions of the material in this section (most notably the illustrations) are taken or adapted from the following online references:

**http://www.math.uncc.edu/~droyster/courses/fall96/math3181/notes/hyprgeom.html**

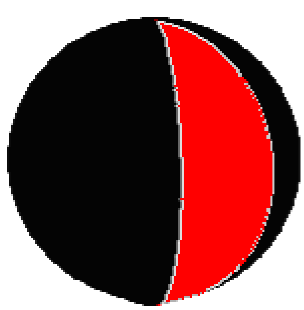
**http://math.rice.edu/~pcmi/sphere/**

**http://www.hps.cam.ac.uk/starry/sphertrig.html**

**http://mathworld.wolfram.com/SphericalTrigonometry.html**

**http://www.math.ubc.ca/~cass/courses/m308-02b/projects/franco/index.htm**

In the plane, there are no interesting polygons with only two sides. This is not true on the sphere. A pair of great circles meets in two antipodal points, and these curves divide the sphere into four regions, each of which has two edges which are semicircles of great circles. The two semicircles bounding such a region form a ***lune*** (pronounced "loon"), or a ***biangle***; the first name reflects the fact that the regions bounded by lunes correspond to the phases of the moon that are visible from the earth at any given time.



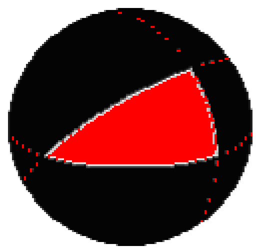
Lunes are fairly simple objects, but they have a few properties that we shall note:

1. The vertices of a lune are antipodal points.
2. The two vertex angles of a lune have equal measures.
3. The areas of the smaller and larger regions bounded by a lune are determined by the measures of these vertex angles.

For the sake of completeness, we should note that the angles are measured using the tangent rays to the semicircles at the two vertex points where the latter meet.

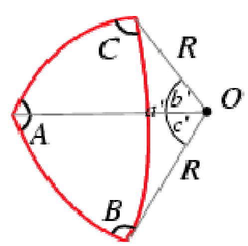
*Spherical triangles*

Spherical triangles are is defined just like planar triangles; they consists of three points which do not lie on a great circle and are called ***vertices***, and three arcs of great circles that join the vertices, which are called the ***sides***. An illustration is given below.



To simplify matters, we shall concentrate on <u>*small triangles*</u>, in which the sides are minor great circle arcs. Most if not all results actually hold for "large" triangles; the derivations are not particularly difficult, but here we are interested in describing the main points of spherical geometry rather than stating and proving the best possible results.

Just as there are six basic measurements associated to a plane triangle, there are also six basic measurements associated to a spherical triangle. In the picture below, they correspond to the ***degree measures of the minor great circle arcs*** joining **A** to **B**, **B** to **C**, and **A** to **C** (analogous to the lengths of the sides), and the ***measures of the vertex angles*** at **A**, **B** and **C**. These vertex angles are measured exactly like the vertex angles for lunes. For example, the vertex angle at **A** is measured by considering the lune formed by the two great semicircles which have **A** as one endpoint and pass through the points **B** and **C**. Of course, similar considerations apply to the other two vertex angles.




The basic geometry and trigonometry of spherical triangles has been worked out fairly completely, and it resembles the theory of ordinary plane triangles. In particular, one has analogs for the following theorems in plane geometry and trigonometry:

1. The Pythagorean Theorem (but the spherical formula is different!).

2. The **SAS, ASA, AAS,** and **SSS** congruence theorems for triangles.

3. The Law of Sines and the Law of Cosines.

4. Standard inequalities involving measurements of triangle parts (*e.g.*, the longer side is opposite the larger angle, the strict Triangle Inequality).

**One major difference between plane and spherical triangles is that we have an AAA congruence theorem for the latter.**

**Theorem 2.** (**AAA** Congruence Theorem) *Suppose that we are given two (small) spherical triangles on the same sphere* $\Sigma$ *with vertices* **A, B, C** *and* **D, E, F.** *If the measures of the vertex angles at* **A, B, C** *are equal to the measures of the vertex angles at* **D, E, F** *respectively, then the lengths of the arcs* **€(AB), €(BC),** *and* **€(AC)** *are equal to the lengths of the arcs* **€(DE), €(EF),** *and* **€(DF)** *respectively.* ■

It is natural to ask why there is such a result for spherical triangles when the analog for plane triangles is completely false, and there is a fairly simple conceptual answer to this question which involves an important relationship between the surface area of the spherical region bounded by a spherical triangle and the sum of the measures of its vertex angles.

The following old textbooks discuss the proofs of the results mentioned above (however, the proofs do not always meet modern standards of rigor, and in some cases may even be incomplete):

W. B. Ford, C. Ammerman, and E. R. Hedrick, ***Solid Geometry.*** Macmillan, New York, 1913. Available online at **http://books.google.com/books?id=zd9EAAAAIAAJ**

H. E. Slaught and N. J. Lennes, ***Solid Geometry: With Problems and Applications*** (Revised Ed.). Allyn and Bacon, Boston, 1919. Available online at **http://books.google.com/books?id=LTIAAAAAYAAJ**

G. A. Wentworth, Solid Geometry (Revised Ed.). Ginn, Boston, 1899. Available online at **http://books.google.com/books?id=CjAAAAAAYAAJ**

The file **http://math.ucr.edu/~res/math133/oldreferences.pdf** gives a few other references to solid geometry textbooks.

*Angle sums and surface area in spherical geometry*

It is intuitively clear that a small spherical triangle with vertices **A, B,** and **C** bounds a closed region in the sphere which is analogous to the closed interior of a plane triangle; in particular, this can be seen from the two previous drawings of spherical triangles. Specifically, the closed "interior" region on the sphere determined by the spherical triangle is the union of the spherical triangle with the spherical region defined by the intersection of the following sets:

1. The sphere itself.
2. The set of all point in space on the same side of the plane **OAB** as **C.**
3. The set of all point in space on the same side of the plane **OAC** as **B.**
4. The set of all point in space on the same side of the plane **OBC** as **A.**

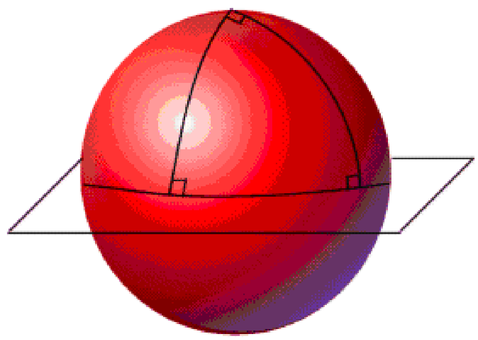
***Problem:*** *What is the area of this closed region?*

The answer is given by the following result due to A. Girard (1595 – 1632).

**Theorem 3.** *Let* **A, B, C** *be the vertices of a spherical triangle as above, and assume that the sphere containing them has radius* $k$. *Let* $\alpha$, $\beta$, $\gamma$ *be the measures of the vertex angles of the spherical triangle above at* **A, B** *and* **C,** *all expressed in radians. Then the angle sum* $\alpha + \beta + \gamma$ *is greater than* $\pi$, *and the area of the closed region bounded by the spherical triangle with these vertices is equal to* $k^2(\alpha + \beta + \gamma - \pi)$.

***Notation.*** The difference $\alpha + \beta + \gamma - \pi$ is called the ***spherical excess*** of the spherical triangle.

Here is an **example** to illustrate the conclusions: Consider the spherical triangle below, which has one vertex at the North Pole and two on the Equator. The measures of the angles at the equatorial vertices are both $90$ degrees, and clearly we can take the measure $E$ of the angle at the polar vertex to be anything between $90°$ and $180°$. The spherical excess of such a triangle (measured in degrees) is then equal to $E$, and the area of the spherical triangle is equal to $k^2 \pi E/180$.

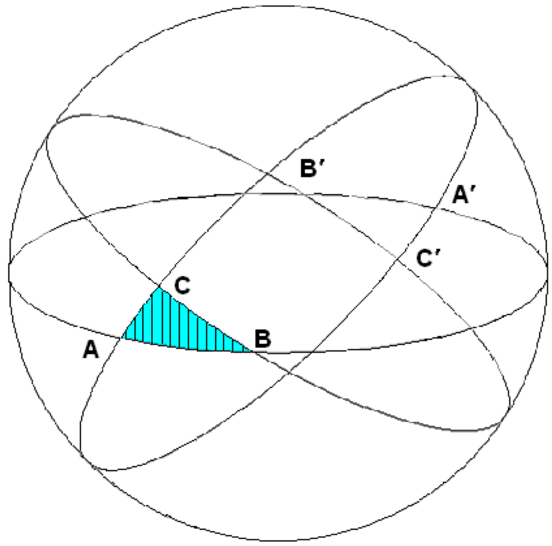


Incidentally, Heron's Formula for the area of a plane triangle in terms of the lengths of the sides has an analog for spherical triangles which is due to S. L'Huillier (1750 – 1840):

$$\tan\left(\tfrac{1}{4}E\right) = \sqrt{\tan\left(\tfrac{1}{2}s\right)\tan\left[\tfrac{1}{2}(s-a)\right]\tan\left[\tfrac{1}{2}(s-b)\right]\tan\left[\tfrac{1}{2}(s-c)\right]}.$$

In this formula $E$ denotes the spherical excess of the triangle, while $a$, $b$, $c$ represent the lengths of the sides opposite vertices **A, B, C** and $s = ½(a + b + c)$.

***Sketch of a proof for Girard's Theorem.*** Consider the special case in which the spherical triangle lies on a closed hemisphere (for example, all points on the equator or the northern hemisphere); as in previous discussions, it is possible to retrieve the general case from such special cases. If the spherical triangle does lie on a closed hemisphere, then the great circles containing the arcs **€(AB), €(BC), €(AC)** split the sphere into eight closed regions as illustrated below:




We know the areas of all the lunes determined by the spherical triangle, and these areas also turn out to be equal to the sums of the areas of the pieces into which these lunes are cut by the various great circles. Algebraic manipulation of these identities yields the area formula stated in the theorem. Further details are given at the following online site:

http://math.rice.edu/~pcmi/sphere/gos4.html#1

Girard's Theorem foreshadowed one of the most important results in non – Euclidean geometry that will be discussed in Section 4 of this unit. ■

## V.2 : Attempts to prove Euclid's Fifth Postulate

We have already noted that questions about Euclid's Fifth Postulate are almost certainly at least as old as the ***Elements*** itself. Apparently the first known attempt to prove this assumption from the others was due to Posidonius (135 – 51 B. C. E.), and the question is discussed at some length in the writings of Proclus from the 5th century. A thorough description of all known attempts to answer this question is beyond the scope of these notes, but we shall note that the writings of Omar Khayyam (Ghiyās od – Dīn Abul – Fatah Omār ibn Ibrāhīm Khayyām Nishābūrī, 1048 – 1122) and Nasireddin/Naṣīr al – Dīn al – Ṭūsī (Muḥammad ibn Muḥammad ibn al – Ḥasan al – Ṭūsī, 1201 – 1274) anticipated some important aspects of the subject, and later work of J. Wallis (1616 – 1703) was also significant in several respects. None of these scholars succeeded in proving the Fifth Postulate, but in many cases they showed that it is logically equivalent to certain other statements that often seem extremely reasonable. For example, in the work of Proclus, the Fifth Postulate is shown to be equivalent to an assumption that ***the distance between two given parallel lines is bounded from above by some constant***, and Wallis showed that the Fifth Postulate is true if ***one can construct triangles that are similar but not necessarily congruent to a given one***.

*New viewpoints and increasing sophistication*

By the end of the 16th century, mathematics had begun to evolve well beyond the classical work of the Greeks and non – European cultures in the Middle East, India and China. This growth accelerated during the 17th century, which is particularly noteworthy for the emergence of coordinate geometry and calculus. Both of these had major implications for geometry. First of all, they answered many difficult problems of classical geometry in a fairly direct fashion. Furthermore, they led to new classes of problems that could be studied effectively and powerful new techniques (compare the citation at the beginning of Unit **II**). During the 18th century mathematics continued to expand in several directions. In particular, mathematicians such as Euler made many striking discoveries about Euclidean geometry that were (apparently) unknown to the Greeks, and in view of the increasing mathematical sophistication of the time it is not surprising that increasingly sophisticated efforts to prove Euclid's Fifth Postulate began to appear. In many cases, the basic idea was to assume this assumption is false and to obtain a contradiction; if this could be done, then one could conclude that the Fifth Postulate was a logical consequence of the other assumptions.

*What if the Fifth Postulate is false?*

> Though this be madness, yet there is method in it.
>
> Shakespeare, ***Hamlet***, Act **2**, Scene **2**, line **206**

Sustained and extensive efforts by mathematicians to prove the Fifth Postulate began to emerge near the end of the 17th century. One of the earliest attempts was due to G. Saccheri (1667 – 1733). His work is particularly noteworthy in his approach; namely, his idea was to show that a contradiction results if one assumes the Fifth Postulate is false, and he went quite far in analyzing what would happen if this were the case. The results indicated that there were two distinct options if one did not necessarily assume the Fifth Postulate; one of them is Euclidean geometry and the other is a system which is like Euclidean geometry in many respects but also has some properties which seem bizarre at first glance. His conclusions were very accurate until the very end, where he dismissed the non – Euclidean alternative as "repugnant to the nature of a straight line." Saccheri's work was not widely known during the 18th century, and A. – M. Legendre (1752 – 1833) independently obtained many of his results as well as some others.

A few 18th century mathematicians drew conclusions that anticipated the breakthroughs of the next century. The 1763 dissertation of G. S. Klügel (1739 – 1812) pointed out mistakes in 28 purported proofs of the Fifth Postulate, and the author expressed doubt that any proof at all was possible. Perhaps the most penetrating insights during this period were due to J. H. Lambert (1728 – 1777). He did not claim to prove the Fifth Postulate, but instead he speculated that the geometry obtained by assuming the negation of Fifth Postulate was the geometry of "a sphere of imaginary radius" (***i.e.,*** the square of the radius is a negative real number). This probably seemed very strange to many of his contemporaries, but the advances of the 19th century show it reflects some very important aspects of non – Euclidean geometry.

Lambert's insights were taken further by F. K. Schweikart (1780 – 1859) and F. A. Taurinus (1794 – 1874). Schweikart developed the alternative explicitly as a subject in its own right and called it ***astral geometry***, speculating that it might be true in "the space of the stars." Taurinus proceeded to derive the formulas of the analytic geometry for the alternative system. These formulas are exactly what one obtains by taking the standard formulas from spherical geometry and trigonometry by substituting an imaginary number for the radius of the sphere; this provided a strong confirmation of Lambert's earlier speculation. Independently, Gauss had discovered the same relationships and become convinced that no mathematical proof of the Fifth Postulate from the other assumptions was possible.

*Statements equivalent to the Fifth Postulate*

We have noted that much of the work on the Fifth Postulate can be viewed as showing that various statements are logically equivalent to that statement. Here is a long but not exhaustive list of theorems in Euclidean geometry that are logically equivalent to the Fifth Postulate.

1. If two lines are parallel to a third line, then they are parallel to each other.
2. The angle sum of a triangle is $180$ degrees.
3. The angle sum of at least one triangle is $180$ degrees.
4. There exists at least one rectangle.
5. There exist two parallel lines that are everywhere equidistant.
6. The distance between two parallel lines is bounded from below by a positive constant.
7. The distance between two parallel lines is bounded from above by a positive constant.
8. If a line meets one of two parallel lines, it meets the other.
9. There exist two similar but noncongruent triangles.
10. The opposite sides of a parallelogram have equal length.
11. Every line containing a point in the interior of an angle must meet at least one ray of the angle.
12. Through a given point in the interior of an angle there is a line that meets both rays of the angle.
13. Given any area function for the closed interiors of triangles, there exist triangles with arbitrarily large areas.
14. If two parallels are cut by a transversal, the alternate interior angles have equal measures.
15. The line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half its length.
16. Given three noncollinear points, there is a circle containing all three of them.
17. The ratio of the circumference of a circle to its diameter is constant.
18. An angle inscribed in a semicircle is a right angle.
19. The Pythagorean Theorem.
20. Given two positive real numbers $a$ and $b$, there is a rectangle whose sides have lengths equal to $a$ and $b$.

21. Through a point not on a given line there passes not more than one parallel to the line.
22. Parallel lines are everywhere equidistant from one another.
23. There exists a quadrilateral whose angle sum is $360$ degrees.
24. Any two parallel lines have a common perpendicular.

In the next two sections of these notes we shall investigate some of these equivalences.

We shall conclude this section with a summary of the results from Euclidean geometry whose proofs do not require the Fifth Postulate.

*How much can one prove **without** the Fifth Postulate?*

Clearly the first step in studying the role of the Fifth Postulate is to understand which results in Euclidean geometry do not depend logically upon that statement. There are numerous examples of proofs from Unit **III** which do not depend upon the Fifth Postulate or an equivalent statement, but there are also other cases where our proofs depend upon these assumptions but it is also possible to give proofs which do not. Therefore we list here some geometric results which hold regardless of whether or not the Fifth Postulate is true; proofs for some of these results are sketched in the exercises. For the sake of convenience, we use the numbering of results from the notes.

**<u>Proposition II.2.4.</u>** *Suppose that* **A, B, C, D** *are four distinct collinear points satisfying the conditions* **A∗B∗D** *and* **B∗C∗D.** *Then* **A∗B∗C** *and* **A∗C∗D** *also hold.*

**<u>Theorem II.2.5.</u>** *Let* **a, b, c** *be three distinct collinear points. Then either* $\mathbf{c} \in (\mathbf{ab}$ *or else* $\mathbf{c} \in (\mathbf{ab}^{\mathbf{OP}}$. *In the first case we have*

$$[\mathbf{ab} = [\mathbf{ac},\ \ [\mathbf{ab}^{\mathbf{OP}} = [\mathbf{ac}^{\mathbf{OP}},\ \ (\mathbf{ab} = (\mathbf{ac},\ \ \textit{and}\ \ (\mathbf{ab}^{\mathbf{OP}} = (\mathbf{ac}^{\mathbf{OP}}.$$

*In the second case we have*

$$[\mathbf{ab} = [\mathbf{ac}^{\mathbf{OP}},\ \ [\mathbf{ab}^{\mathbf{OP}} = [\mathbf{ac},\ \ (\mathbf{ab} = (\mathbf{ac}^{\mathbf{OP}},\ \ \textit{and}\ \ (\mathbf{ab}^{\mathbf{OP}} = (\mathbf{ac}.$$

**<u>Proposition II.2.8.</u>** *Let* **M** *denote either a line* **L** *in a plane* **P** *or a plane* **Q** *in space. Then the following hold:*

1. *If* **A** *and* **B** *are on the same side of* **M** *and* **B** *and* **C** *are on the same side of* **M,** *then* **A** *and* **C** *are on the same side of* **M.**
2. *If* **A** *and* **B** *are on the same side of* **M** *and* **B** *and* **C** *are on opposite sides of* **M,** *then* **A** *and* **C** *are on the same side of* **M.**
3. *If* **A** *and* **B** *are on opposite sides of* **M** *and* **B** *and* **C** *are on the same side of* **M,** *then* **A** *and* **C** *are on opposite sides of* **M.**

**<u>Lemma II.2.10.</u>** *Let* **L** *be a line in the plane, and let* **M** *be a line in the plane which meets* **L** *in exactly one point. Then* **M** *contains points on both sides of* **L.**

**<u>Proposition II.2.11.</u>** *Let* **L** *be a line in the plane, let* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *be the two half – planes determined by* **L,** *and let* **M** *be a line in the plane which meets* **L** *in exactly one point. Then each of the intersections* $\mathbf{H}_1 \cap \mathbf{M}$ *and* $\mathbf{H}_2 \cap \mathbf{M}$ *is an open ray.*

**<u>Proposition II.2.12.</u>** *Let* **L** *be a line in the plane, let* **M** *be a line in the plane which meets* **L** *in exactly one point* **A,** *and let* **B** *and* **C** *be two other points on* **M.** *Then* **B** *and* **C** *lie on the same side of the line* **L** *if either* **A∗C∗B** *or* **A∗B∗C** *is true, and they lie on opposite sides of the line* **L** *if* **B∗A∗C** *is true.*

**<u>Theorem II.2.13. (Pasch's "Postulate")</u>** *Suppose we are given* **△ABC** *and a line* **L** *in the same plane as the triangle such that* **L** *meets the open side* **(AB)** *in exactly one point. Then either* **L** *passes through* **C** *or else* **L** *has a point in common with* **(AC)** *or* **(BC).**

**<u>Proposition II.3.1.</u>** *Let* **A** *and* **B** *be distinct points, and let* $x$ *be a positive real number. Then there is a unique point* **Y** *on the open ray* **(AB** *such that* $d$**(A, Y)** $=$ $x$. *Furthermore, we have* **A∗Y∗B** *if and only if* $x$ $<$ $d$**(A, B),** *and likewise we have* **A∗B∗Y** *if and only if* $x$ $>$ $d$**(A, B).**

**<u>Theorem II.3.5.</u>** <u>(Crossbar Theorem)</u> *Let* **A, B, C** *be noncollinear points in* $\mathbb{R}^2$, *and let* **D** *be a point in the interior of* **∠CAB.** *Then the segment* **(BC)** *and the open ray* **(AD** *have a point in common.*

**<u>Proposition II.3.6.</u>** <u>(Trichotomy Principle)</u> *Let* **A** *and* **B** *be distinct points in* $\mathbb{R}^2$, *and let* **C** *and* **D** *be two points on the same side of* **AB**. *Then exactly one of the following is true:*

**(1)** **D** *lies on* **(BC** (*equivalently, the open rays* **(BC** *and* **(BD** *are equal*).

**(2)** **D** *lies in* **Int ∠ABC.**

**(3)** **C** *lies in* **Int ∠ABD.**

**<u>Proposition II.3.7. (Vertical Angle Theorem</u>)** *Let* **A, B, C, D** *be four distinct points such that* **A∗X∗C** *and* **B∗X∗D.** *Then* **|∠AXB|** $=$ **|∠CXD|.**

**<u>Theorem II.3.8.</u>** *Let* **A, B, C, D** *be distinct coplanar points, and suppose that* **C** *and* **D** *lie on the same side of* **AB.** *Then* **|∠CAB|** $<$ **|∠DAB|** *is true if and only if* **C** *lies in the interior of* **∠DAB.**

**<u>Theorem II.4.1. (Isosceles Triangle Theorem)</u>** *In* **△ABC,** *one has* $d$**(A, B)** $=$ $d$**(A, C)** *if and only if* **|∠ABC|** $=$ **|∠ACB|.**

**<u>Corollary II.4.2.</u>** *In* **△ABC,** *one has* $d$**(A, B)** $=$ $d$**(A, C)** $=$ $d$**(B, C)** *(the triangle is **equilateral**) if and only if one has* **|∠ABC|** $=$ **|∠ACB|** $=$ **|∠BAC|** *(the triangle is **equiangular**)*.

**<u>Proposition III.1.1.</u>** *Let* **A, B, C** *be noncollinear points, and suppose that* **E** *is a point such that* **E∗A∗C** *holds. Then* **AB ⊥ AC** *if and only if* **|∠EAB|** $=$ **|∠CAB|.**

**<u>Corollary III.1.2.</u>** *Let* **A, B, C** *be noncollinear points, and suppose that* **D** *and* **E** *are points such that both* **E∗A∗C** *and* **B∗A∗D** *hold. Then* **AB ⊥ AC** *if and only if*

$$|\angle \mathbf{CAB}| = |\angle \mathbf{EAB}| = |\angle \mathbf{EAD}| = |\angle \mathbf{DAC}| = 90°.$$

**Proposition III.1.3.** *Let* **L** *be a line, let* **A** *be a point of* **L,** *and let* **P** *be a plane containing* **L.** *Then there is a unique line* **M** *in* **P** *such that* $\mathbf{A} \in \mathbf{M}$ *and* $\mathbf{L} \perp \mathbf{M}$.

**Proposition III.1.4.** *Let* **L** *be a line, and let* **A** *be a point* ***not*** *on* **L.** *Then there is a unique line* **M** *such that* $\mathbf{A} \in \mathbf{M}$ *and* $\mathbf{L} \perp \mathbf{M}$.

**Corollary III.1.5.** *Suppose that* **L, M** *and* **N** *are three lines in the plane* **P** *such that we have* $\mathbf{L} \perp \mathbf{M}$ *and* $\mathbf{M} \perp \mathbf{N}$. *Then we also have* $\mathbf{L} \,||\, \mathbf{N}$.

**Proposition III.1.7.** *Let* **A** *and* **B** *be distinct points, let* **P** *be a plane containing them, suppose that* **D** *is the midpoint of* **[AB],** *and let* **M** *be the unique perpendicular to* **AB** *at* **D** *in the plane* **P.** *Then a point* $\mathbf{X} \in \mathbf{P}$ *lies on* **M** *if and only if* $d(\mathbf{X}, \mathbf{A}) = d(\mathbf{X}, \mathbf{B})$.

**Theorem III.1.8.** *Suppose we are given a plane* **P** *and a line* **L** *not contained in* **P** *such that* **L** *and* **P** *meet at the point* **x.** *Suppose further that there are two distinct lines* **M** *and* **N** *in* **P** *such that* **x** *lies on both and* **L** *is perpendicular to both* **M** *and* **N.** *Then* **L** *is perpendicular to* **P.**

**Theorem III.1.9.** *If* **P** *is a plane and* **x** *is a point in space, then there is a unique line through* **x** *which is perpendicular to* **P.**

**Theorem III.1.12.** *If* **L** *is a line and* **x** *is a point in space, then there is a unique plane through* **x** *which is perpendicular to* **L.**

**Theorem III.2.1. (Exterior Angle Theorem)** *Suppose we are given triangle* $\triangle\mathbf{ABC}$, *and let* **D** *be a point such that* $\mathbf{B}*\mathbf{C}*\mathbf{D}$. *Then* $|\angle \mathbf{ACD}|$ *is greater than both* $|\angle \mathbf{ABC}|$ *and* $|\angle \mathbf{BAC}|$.

**Corollary III.2.2.** *If* $\triangle\mathbf{ABC}$ *is an arbitrary triangle, then the sum of any two of the angle measures* $|\angle \mathbf{ABC}|$, $|\angle \mathbf{BCA}|$ *and* $|\angle \mathbf{CAB}|$ *is less than* $180°$. *Furthermore, at least two of these angle measures must be less than* $90°$.

**Corollary III.2.3.** *Suppose we are given triangle* $\triangle\mathbf{ABC}$, *and assume that the two angle measures* $|\angle \mathbf{BCA}|$ *and* $|\angle \mathbf{CAB}|$ *are less than* $90°$. *Let* $\mathbf{D} \in \mathbf{AC}$ *be such that* **BD** *is perpendicular to* **AC.** *Then* **D** *lies on the open segment* **(AC).**

**Corollary III.2.4.** *Suppose we are given triangle* $\triangle\mathbf{ABC}$. *Then at least one of the following three statements is true:*

(1) *The perpendicular from* **A** *to* **BC** *meets the latter in* **(BC).**
(2) *The perpendicular from* **B** *to* **CA** *meets the latter in* **(CA).**
(3) *The perpendicular from* **C** *to* **AB** *meets the latter in* **(AB).**

**Theorem III.2.5.** *Given a triangle* $\triangle\mathbf{ABC}$, *we have* $d(\mathbf{A}, \mathbf{C}) > d(\mathbf{A}, \mathbf{B})$ *if and only if we have* $|\angle \mathbf{ABC}| > |\angle \mathbf{ACB}|$.

**Theorem III.2.6. (Classical Triangle Inequality)** *In* $\triangle$**ABC**, *we have the inequality* $d(\mathbf{A}, \mathbf{C}) < d(\mathbf{A}, \mathbf{B}) + d(\mathbf{B}, \mathbf{C})$.

**Proposition III.2.10. (Half of the Alternate Interior Angle Theorem)** *Suppose we are given the setting and notation above. If the measures of one pair of alternate interior angles are equal, then the lines* **L** *and* **M** *are parallel.*

**Corollary III.2.15. (AAS Triangle Congruence Theorem)** *Suppose we have two ordered triples of noncollinear points* **(A, B, C)** *and* **(D, E, F)** *satisfying the conditions* $d(\mathbf{B}, \mathbf{C}) = d(\mathbf{E}, \mathbf{F})$, $|\angle \mathbf{ABC}| = |\angle \mathbf{DEF}|$, *and* $|\angle \mathbf{CAB}| = |\angle \mathbf{FDE}|$. *Then we have* $\triangle \mathbf{ABC} \cong \triangle \mathbf{DEF}$.

The preceding result turns out to be particularly important if we do not assume the Fifth Postulate (for example, see the proof of Proposition V.2.1 below).

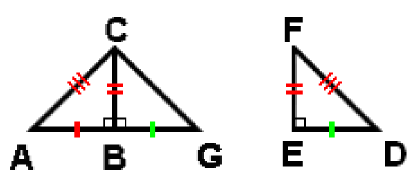
**Proposition III.3.1.** *Suppose that* **A, B, C** *and* **D** *form the vertices of a convex quadrilateral. Then the open diagonal segments* **(AC)** *and* **(BD)** *have a point in common.*

**Reminder. We are NOT necessarily claiming that the proofs of these results in the notes do not use the Fifth Postulate or an equivalent statement. Frequently it is necessary to give a new (usually longer and more complicated) proof, and *the arguments which use techniques from linear algebra must ALWAYS be replaced by synthetic approaches*.**

Finally, here is a result (***the hypotenuse – side congruence theorem for right triangles***) which was not previously stated in the notes but is extremely useful. The proof is also a simple illustration of the synthetic methods in this unit.

**Proposition V.2.1. (HS Right Triangle Congruence Theorem)** *Suppose we have two ordered triples of noncollinear points* **(A, B, C)** *and* **(D, E, F)** *satisfying the conditions* $|\angle \mathbf{ABC}| = |\angle \mathbf{DEF}| = 90°$, $d(\mathbf{A}, \mathbf{C}) = d(\mathbf{D}, \mathbf{F})$, *and* $d(\mathbf{B}, \mathbf{C}) = d(\mathbf{E}, \mathbf{F})$. *Then we have* $\triangle \mathbf{ABC} \cong \triangle \mathbf{DEF}$.

***Proof.*** Let **G** be the unique point on $(\mathbf{BA}^{\mathbf{OP}}$ such that $d(\mathbf{B}, \mathbf{G}) = d(\mathbf{D}, \mathbf{E})$.




Since $\angle \mathbf{GBC}$ is a right angle we have $\triangle \mathbf{GBC} \cong \triangle \mathbf{DEF}$ by SAS. Therefore we also have $d(\mathbf{G}, \mathbf{C}) = d(\mathbf{D}, \mathbf{F}) = d(\mathbf{A}, \mathbf{C})$, which means that $\triangle \mathbf{AGC}$ is isosceles and hence we also have $|\angle \mathbf{CAB}| = |\angle \mathbf{CGB}| = |\angle \mathbf{FDE}|$. We can now apply AAS to conclude that $\triangle \mathbf{ABC} \cong \triangle \mathbf{DEF}$. ■

The exercises for this section contain synthetic proofs for several of these results which do not use the Fifth Postulate or an equivalent statement. In the next section we shall describe still further results that hold regardless of whether the Fifth Postulate or equivalent statements are assumed to be true or false.

*Intuitive and experimental questions*

Before proceeding to the next phase of the mathematical discussion, we shall mention some ways in which the Fifth Postulate and its equivalent statements are intrinsically more complex than the other assumptions in the ***Elements***. It seems likely (in fact, almost certain) that some of these were recognized at or before Euclid's time.

We have noted that Playfair's Postulate is logically equivalent to Euclid's Fifth Postulate but is formally much simpler to state. However, there are some immediate questions whether Playfair's Postulate actually "reflects physical reality." The key issues are summarized in a passage on page **123** in the book, ***Mathematics: The Science of Patterns***, by K. Devlin (Owl Books, 1996, ISBN: 0–805–07344–2):

> Suppose you drew a line on a sheet of paper and marked a point not on the line. You are now faced with the task of showing that there is one and only one parallel to the given line that passes through the chosen point. But there are obvious difficulties here. For one, no matter how fine the point of your pencil, the lines you draw still have a definite thickness, and how do you know where the [supposedly] actual ***lines*** are? Second, in order to check that your second line is in fact parallel to the first, you would have to extend both lines indefinitely, which is [physically] not possible. Certainly, you can draw many lines through the given point that do not meet the given line ***on the paper***.

In some sense, the difference between Euclid's Fifth Postulate and Playfair's Postulate is that the former assumes gives a condition that two lines will eventually meet at some possibly remote location, but the latter assumes there are lines that will never meet. If one prefers to avoid questions whether two lines might meet at locations that are effectively physically inaccessible, then the option of assuming an equivalent statement about a bounded portion of space may seem promising. The discussion in Devlin's book also addresses this.

> Thus, Playfair's Postulate is not really suitable for experimental verification. How about the triangle postulate [namely, the angle sum of some triangle is **180** degrees]? Certainly, verifying this postulate does not require extending the lines indefinitely; it can all be done "on the paper." Admittedly, it is likely that no one has any strong intuition concerning the angle sum of a triangle being **180** degrees, the way we do about the existence of unique parallels, but since the two statements are entirely equivalent, the absence of any supporting intuition does not affect the validity of the triangle approach.

If we want to test the statement about angle sums experimentally, we run into immediate problems. First of all, the unavoidability of experimental errors means it is effectively impossible to draw any firm conclusions that the angle sum is ***exactly*** **180** degrees. In contrast, it is conceivable that experimental measurements could show that the angle sum is ***NOT*** equal to **180** degrees, with the deviation exceeding any possible experimental error. There are frequently repeated assertions that Gauss actually tried to carry out such an experiment but his results were inconclusive because the value of **180** degrees was within the expected margin of error. However, there is no hard evidence to confirm such stories.

# V.3 : Neutral geometry

In this section we shall investigate some of the logical equivalences in the list from the previous section. These will play an important role in Section 4.

We have noted that a great deal of work was done in the 17th and 18th century to study classical geometry without using Euclid's Fifth Postulate; early in the 19th century this subject was called ***absolute geometry***, but in modern texts it is generally known as ***neutral geometry***. In this section we shall develop some aspects of this subject more explicitly than in the preceding section. We shall begin with a formal definition of the setting for neutral geometry.

<u>**Definition.**</u> A ***neutral plane*** is given by data $(\mathbf{P}, \mathcal{L}, d, \mu)$ which satisfy all the axioms of Unit $\mathbf{II}$ except (possibly) Playfair's Postulate or an equivalent statement such as Euclid's Fifth Postulate. Usually we simply denote a neutral plane by its underlying set of points $\mathbf{P}$.

In this setting, the efforts to prove the Fifth Postulate can be restated as follows:

<u>**INDEPENDENCE PROBLEM FOR THE FIFTH POSTULATE.**</u> *If* $\mathbf{P}$ *is a neutral plane, is Playfair's Postulate* $(\mathbf{P}-\mathbf{0})$ *true in* $\mathbf{P}$?

Before considering this question, we need to summarize some results from classical geometry that remain true in a neutral plane. It is important to note that ***all proofs for neutral planes must be done synthetically*** because Playfair's Postulate is essentially built into the analytic approach to Euclidean geometry.

*The Saccheri – Legendre Theorem*

One of the cornerstones of neutral and non – Euclidean geometry is the study of the following issue:

<u>**ANGLE SUMS OF TRIANGLES.**</u> *Given a triangle* $\triangle\mathbf{ABC}$, *what can we say about the angle sum* $|\angle\mathbf{ABC}| + |\angle\mathbf{BCA}| + |\angle\mathbf{CAB}|$ *and what geometric information does it carry?*

We know that the angle sum in Euclidean geometry is always $180°$, and as noted in the preceding section this fact is logically equivalent to the Fifth Postulate. On the other hand, we have also seen that the angle sum in spherical geometry is always greater than and that the difference between these quantities is proportional to the area of a spherical triangle. In any case, the angle sum of a triangle was a central object of study in 17th and 18th century efforts to prove the Fifth Postulate.

Most of the arguments below are similar to proofs in high school geometry, with extra attention to questions about order and separation. However, at several points we need the following properties of real numbers.

<u>**Archimedean Law.**</u> *Suppose that* $b$ *and* $a$ *are positive real numbers. Then there is a positive integer* $n$ *such that* $na > b$. ■

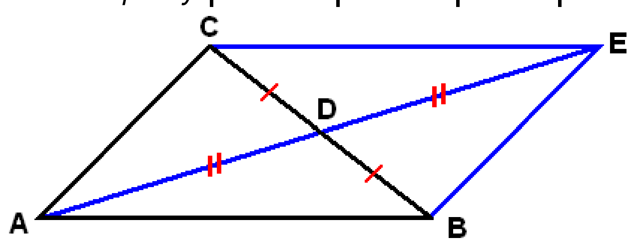
**Immediate consequence.** *If* $h$ *and* $k$ *are positive real numbers, then there is a positive integer* $n$ *such that* $h/2^n < k$.■

The second statement is proven in the exercises. One informal way of seeing the first statement is to note that the positive number $b/a$ can be written as $x + y$, where $x$ is a nonnegative integer and $y$ lies in the half — open interval $[0, 1)$; one can then take $n = x + 1$.

Our first result on angle sums is a result on finding new triangles with the same angle sums as a given one.

**Proposition 1.** *Suppose that* **A, B, C** *are noncollinear points in the neutral plane* **P.** *Then there exist noncollinear points* **A′, B′, C′** *such that the following hold*:

1. *The angle sums of* △**ABC** *and* △**A′B′C′** *are equal; in other words, we have* $|\angle ABC| + |\angle BCA| + |\angle CAB| = |\angle A'B'C'| + |\angle B'C'A'| + |\angle C'A'B'|$.
2. *We also have the inequality* $|\angle C'A'B'| \leq \frac{1}{2}|\angle CAB|$.
3.




***Proof.*** Let **D** be the midpoint of **[BC]**, and let **E** be a point on **(AD** satisfying **A∗D∗E** and $d(\mathbf{A}, \mathbf{E}) = 2d(\mathbf{A}, \mathbf{D})$; it follows that $d(\mathbf{A}, \mathbf{D}) = d(\mathbf{D}, \mathbf{E})$. We then have △**CDA** ≅ △**BDE** by SAS, so it follows that $|\angle CAE| = |\angle AEB|$ and $|\angle ACB| = |\angle CBE|$.

**CLAIM:** △**EAB** ***has the same angle sum as*** △**CAB.** — As in the proof of the Exterior Angle Theorem, we know that **E** lies in the interior of ∠**CAB.** Therefore we have $|\angle CAB| = |\angle CAE| + |\angle EAB|$. Since **C** lies on **(BD** and **D** lies in the interior of ∠**ABE,** we also have $|\angle ABE| = |\angle ABC| + |\angle CBE|$. Combining this with the triangle congruence from the previous paragraph, we see that

$$|\angle EAB| + |\angle ABE| + |\angle BEA| =$$
$$|\angle EAB| + |\angle EBC| + |\angle ABC| + |\angle BEA| =$$
$$|\angle EAB| + |\angle ACB| + |\angle ABC| + |\angle CAE| =$$
$$|\angle BCA| + |\angle ACB| + |\angle ABC|.$$

It follows by a similar argument (reversing the roles of **B** and **C**) that

$$|\angle ACE| + |\angle CEA| + |\angle EAC| =$$
$$|\angle ABC| + |\angle BCA| + |\angle CAB|.$$

In other words, both △**EAC** and △**EAB** have the same angle sum as △**ABC.**

Since $|\angle CAB| = |\angle CAE| + |\angle EAB|$ and the two summands in the right hand expression are positive, at least one of them is less than or equal to $\frac{1}{2}|\angle CAB|$. Depending upon whether $\angle CAE$ or $\angle EAB$ has this property, take $\triangle A'B'C'$ to be $\triangle AEC$ or $\triangle ABE$. ■

<u>**Corollary 2.**</u> *If* $\varepsilon > 0$ *is a positive real number, then there is a triangle* $\triangle A'B'C'$ *which has the same angle sum as* $\triangle ABC$ *but* $|\angle C'A'B'| < \varepsilon$.

<u>***Proof.***</u> Repeated application of Proposition $1$ shows that for each positive integer $n$ there is a triangle $\triangle A_nB_nC_n$ with the same angle sum as $\triangle ABC$ but such that $|\angle C_nA_nB_n| \leq |\angle CAB|/2^n$. Since we know that the right hand side is less than $\varepsilon$ for $n$ sufficiently large, the corollary follows. ■

The preceding results allow us to prove half of the usual Euclidean theorem on angle sums:

<u>**Theorem 3 (Saccheri – Legendre Theorem)**</u> *If* **A, B, C** *are noncollinear points in a neutral plane* **P**, *then* $|\angle ABC| + |\angle BCA| + |\angle CAB| \leq 180°$.

<u>***Proof.***</u> Suppose we have a triangle $\triangle ABC$ for which the angle sum is strictly greater than $180°$, and write $|\angle ABC| + |\angle BCA| + |\angle CAB| = 180° + \delta°$, where $\delta$ is positive. By Corollary $2$ there is a triangle $\triangle A'B'C'$ which has the same angle sum as $\triangle ABC$ but also satisfies $|\angle C'A'B'| < \frac{1}{2}\delta$. It then follows that

$$|\angle B'C'A'| + |\angle A'B'C'| > 180° + \tfrac{1}{2}\delta° > 180°.$$

On the other hand, by a corollary to the Exterior Angle Theorem we also know that the sum of the measures of two vertex angles is always less than $180°$, so we have a contradiction. The problem arises from our assumption that the angle sum of the original triangle is strictly greater than $180°$, and therefore we conclude that the angle sum is at most $180°$. ■

<u>**Corollary 4.**</u> *If* **A, B, C, D** *are the vertices of a convex quadrilateral in a neutral plane* **P**, *then* $|\angle ABC| + |\angle BCD| + |\angle CDA| + |\angle DAB| \leq 360°$.

<u>***Proof.***</u> The idea is standard; we slice the quadrilateral into two triangles along a diagonal (in the drawing below, the diagonal is **[AC]**).


By the definition of a convex quadrilateral we know that **A** lies in the interior of $\angle BCD$ and **C** lies in the interior of $\angle DAB$, so that $|\angle BCD| = |\angle ACD| + |\angle ACB|$ and likewise $|\angle DAB| = |\angle CAD| + |\angle CAB|$. The Saccheri – Legendre Theorem implies that

$$|\angle ABC| + |\angle BCA| + |\angle CAB| \leq 180°$$

$$|\angle ADC| + |\angle DCA| + |\angle DAB| \leq 180°$$

and if we combine these with the sum identities in the preceding sentence we obtain

$$|\angle ABC| + |\angle BCD| + |\angle CDA| + |\angle DAB| =$$

$$|\angle ABC| + |\angle BCA| + |\angle CAB| + |\angle ADC| + |\angle DCA| + |\angle DAB| \leq$$

$$180° + 180° = 360°$$

which is the statement of the corollary.■

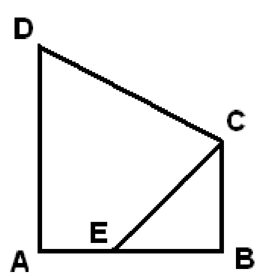
In Section **1** we noted that the angle sum of a triangle is always greater than **180°** degrees in spherical geometry. A unified perspective on neutral and spherical geometry will be discussed in Section **5.**

*Rectangles in neutral geometry*

Rectangles are fundamentally important in both the synthetic and the analytic approaches to Euclidean geometry, so it is not surprising that rectangles and near – rectangles also play an important role in neutral geometry. Before we define the near – rectangles that are studied in neutral geometry, it will be convenient to have a simple criterion for recognizing certain special convex quadrilaterals.

**Proposition 5.** *Let* **A, B, C, D** *be four points in a neutral plane* **P** *such that no three are collinear and* **AB** *is perpendicular to* **BC** *and* **AD.** *If* **C** *and* **D** *lie on the same side of* **AB,** *then* **A, B, C, D** *form the vertices of a convex quadrilateral.*

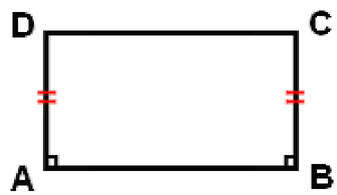
***Proof.*** The lines **BC** and **AD** are parallel since they are perpendicular to the same line (and they are unequal because the four given points are noncollinear). If we combine this with the condition in the second sentence of the proposition, we see that it is only necessary to prove that **A** and **B** lie on the same side of **CD,** so let us suppose this is false. In this case it follows that the segment **(AB)** and the line **CD** have a point **E** in common. We shall use this to derive a contradiction.




Since **C** and **D** lie on the same side of **AB,** it follows that the rays **[EC** and **[ED** are equal (in the picture it appears that the points are not collinear, but this is not a problem since we are trying to derive a contradiction). Therefore we have $|\angle AED| + |\angle CEB| = 180°$. On the other hand, two applications of the Exterior Angle Theorem imply that $|\angle AED| = |\angle AEC| > |\angle CEB| = 90°$ and $|\angle CEB| = |\angle DEB| > |\angle DAB| = 90°$, which in turn implies that $|\angle AED| + |\angle CEB| > 180°$. Thus we have a contradiction; the source of the contradiction was the assumption that **A** and **B** do not lie on the same side of **CD,** and therefore **A** and **B** must lie on the same side of **CD;** as noted before, this is what we needed to complete the proof. ■

The following analogs of rectangles in neutral geometry were studied extensively by Saccheri, but they also appear in earlier mathematical writings of Omar Khayyam.

<u>**Definition.**</u> Let **A, B, C, D** be four points in a neutral plane **P** such that no three are collinear. We shall say that these points ***form the vertices of a <u>Saccheri quadrilateral</u> with base*** **AB** provided that **(1)** the line **AB** is perpendicular to **BC** and **AD,** **(2)** the points **C** and **D** lie on the same side of **AB,** **(3)** the lengths of the sides **[BC]** and **[AD]** are equal — in other words, we have $d(\mathbf{A}, \mathbf{D}) = d(\mathbf{B}, \mathbf{C})$. In some books and articles, such a figure is called an ***isosceles birectangle.***




By the previous proposition we know that **A, B, C, D** are the vertices of a convex quadrilateral, and we say that this quadrilateral **☐ABCD** is a ***Saccheri quadrilateral*** with base **AB. —** The reason for considering Saccheri quadrilaterals is that it is always possible to construct such figures in a neutral plane, and in fact if $p$ and $q$ are arbitrary positive real numbers then there is a Saccheri quadrilateral **☐ABCD** with base **AB** such that $d(\mathbf{A}, \mathbf{D}) = d(\mathbf{B}, \mathbf{C}) = p$ and $d(\mathbf{A}, \mathbf{B}) = q$ (the proof is left to the exercises).

Of course, a rectangle in Euclidean geometry is a Saccheri quadrilateral, and the next result describes some common properties of Euclidean rectangles that also hold for Saccheri quadrilaterals in neutral geometry.

<u>**Proposition 6.**</u> *If* **A, B, C, D** *are the vertices of a Saccheri quadrilateral with base* **AB,** *then* $d(\mathbf{A}, \mathbf{C}) = d(\mathbf{B}, \mathbf{D})$ *and* $|\angle\mathbf{CDA}| = |\angle\mathbf{DCB}| \leq 90°$. *Furthermore, the line joining the midpoints of* **[AB]** *and* **[CD]** *is perpendicular to both* **AB** *and* **CD.**

A proof of this result is sketched in the exercises. ■

The second part of the previous result implies that the line joining the midpoints of the top and base split the Saccheri quadrilateral into to near – rectangles with a different definition.

<u>**Definition.**</u> Let **A, B, C, D** be four points in a neutral plane **P** such that no three are collinear. We shall say that these points ***form the vertices of a <u>Lambert quadrilateral</u>*** provided three of the four lines **AB, BC, CD, DA** are perpendicular to each other (hence there are right angles at three of the four vertices).

In this case it is also straightforward to see that **A, B, C, D** are the vertices of a convex quadrilateral. Suppose, say, that we have right angles at **A, B** and **C.** As in the case of Saccheri quadrilaterals we know that **AD** is parallel to **BC,** but now we also know that **AB** and **CD** are parallel because they are both perpendicular to **BC.** Predictably, under these conditions we say that the quadrilateral **☐ABCD** is a ***Lambert quadrilateral. —*** It follows that if **☐XYZW** is a Saccheri quadrilateral with base **XY,** then the line joining the midpoints of **[XY]** and **[ZW]** splits □**XYZW** into two Lambert quadrilaterals (this is shown in one of the exercises).

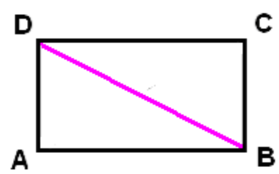
It is also fairly easy to construct Lambert quadrilaterals in a neutral plane. In fact, if $p$ and $q$ are arbitrary positive real numbers then there is a Lambert quadrilateral □**ABCD** with right angles at **A, B,** and **C** such that $d(\mathbf{A}, \mathbf{D}) = p$ and $d(\mathbf{A}, \mathbf{B}) = q$ (the proof is again left to the exercises). By Corollary $4$ we know that $|\angle\mathbf{CDA}| \leq 90°$.

Having defined types of near – rectangles that exist in every neutral plane, we can now give a neutral – geometric definition of "genuine" rectangles:

**<u>Definition.</u>** Let **A, B, C, D** be four points in a neutral plane **P** such that no three are collinear. We shall say that these points ***form the vertices of a <u>rectangle</u>*** provided the four lines **AB, BC, CD, DA** are perpendicular to each other at **A, B, C** and **D.** As before, the points **A, B, C, D** are the vertices of a convex quadrilateral, and we say that the quadrilateral □**ABCD** is a ***rectangle.***

**Since we do not have Playfair's Postulate at our disposal, we must be careful about not using results from Euclidean geometry which depend upon this postulate when we prove theorems about rectangles in neutral geometry,** and frequenty we need new synthetic proofs for extremely familiar facts. Here are a few examples.

**<u>Theorem 7.</u>** *If* **A, B, C, D** *are the vertices of a rectangle, then* $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{C}, \mathbf{D})$ *and* $d(\mathbf{A}, \mathbf{D}) = d(\mathbf{B}, \mathbf{C})$. *Furthermore, we have* $\triangle\mathbf{DAB} \cong \triangle\mathbf{BCD}$, *and the angle sums for these triangles are equal to* $180°$.




***<u>Proof.</u>*** In order to simplify some of the algebraic manipulations, it is helpful to denote various angle measures by letters:

$$|\angle\mathbf{ADB}| = \alpha, \quad |\angle\mathbf{DBA}| = \beta, \quad |\angle\mathbf{DBC}| = \gamma, \quad |\angle\mathbf{BDC}| = \delta$$

Since **D** and **B** lie in the interiors of $\angle\mathbf{CBA}$ and $\angle\mathbf{CDA}$ respectively, it follows that

$$\alpha + \delta = 90° = \beta + \gamma.$$

On the other hand, the Saccheri – Legendre Theorem implies and the perpendicularity conditions imply that $\alpha + \beta \leq 90°$ and $\gamma + \delta \leq 90°$. These imply that the sum of $\alpha, \beta, \gamma, \delta$ is less than or equal to $180°$, while the displayed equations imply that the sum of these four numbers is equal to $180°$. If either of the inequalities were strict, then the sum would be strictly less than $180°$, and hence we have a pair of equations $\alpha + \beta = 90° = \gamma + \delta$. Thus we have a system of two linear equations for $\alpha, \beta, \gamma, \delta$, and the solutions of this system are given by $\alpha = \beta$ and

$\delta = \gamma$. The conclusion about angle sums for the two triangles $\triangle$**DAB** and $\triangle$**BCD** follows immediately from this.

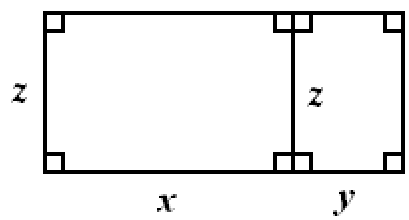
Furthermore, it also follows that $\triangle$**DAB** $\cong$ $\triangle$**BCD** by **SAS.** The remaining conclusions $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{C}, \mathbf{D})$ and $d(\mathbf{A}, \mathbf{D}) = d(\mathbf{B}, \mathbf{C})$ follow from this triangle congruence. ■

**IMPORTANT NOTE.** Although we have defined the concept of a rectangle for an arbitrary neutral plane, ***we do not necessarily know if there are any rectangles at all in a given neutral plane* P *unless we know that Playfair's Postulate holds in* P.** The logical relationship between the Euclidean Parallel Postulate and the existence of rectangles was a central point in the writings of A. – C. Clairaut (1713 – 1765) on classical Euclidean geometry. It turns out that the existence of even one rectangle in **P** has extremely strong consequences, most of which arise from the following result.

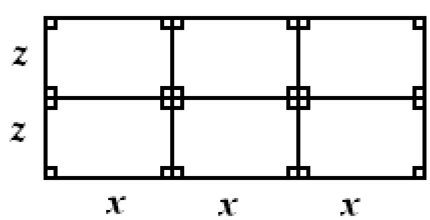
**Theorem 8.** *Suppose there is at least one rectangle in a given neutral plane* **P.** *Then for every pair of positive real numbers* $p$ *and* $q$ *there is a rectangle* ☐**ABCD** *such that* $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{C}, \mathbf{D}) = p$ *and* $d(\mathbf{A}, \mathbf{D}) = d(\mathbf{B}, \mathbf{C}) = q$.

The proof of this theorem is fairly long and has several steps.

1. A splicing construction, which shows if there is a rectangle whose sides have dimensions $x$ and $z$ and a rectangle whose sides have dimensions $y$ and $z$, then there is a rectangle whose sides have dimensions $(x+y)$ and $z$.

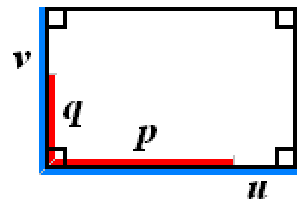



2. Repeated application of the splicing construction to show that if there is a rectangle whose sides have dimensions $x$ and $z$ and we are given positive integers $m$ and $n$, then there is a rectangle whose sides have dimensions $mx$ and $nz$. In the drawing below, $m = 3$ and $n = 2$.

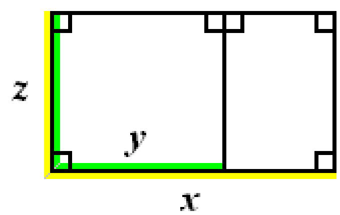



3. Combining the previous two steps with the Archimedean Property of real numbers to show that if a rectangle exists, then there is a rectangle whose sides

have dimensions $u$ and $v$, where $u > p$ and $v > q$.




4. A trimming – down construction, which shows that if there is a rectangle whose sides have dimensions $x$ and $z$ and $y$ is a positive number less than $x$, then there is a rectangle whose sides have dimensions $y$ and $z$. Two applications of this combine with the third step to prove Theorem $8$.




The proofs for several of these steps are quite lengthy in their own right. Therefore we shall merely note that the argument is described on pages $88 - 90$ of the previously cited text by Wallace and West. ■

*The All – or – Nothing Theorem for angle sums*

The preceding result on rectangles has an immediate consequence for angle sums of triangles.

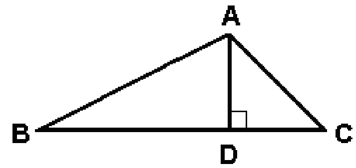
**Theorem 9.** *If a rectangle exists in a neutral plane* **P**, *then every right triangle in* **P** *has an angle sum equal to* $180°$.

***Proof.*** Suppose we are given right triangle $\triangle$**ABC** with a right angle at **B.** By the preceding result there is a rectangle ☐**WXYZ** such that $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{W}, \mathbf{X})$ and $d(\mathbf{B}, \mathbf{C}) = d(\mathbf{X}, \mathbf{Y})$. By SAS we have $\triangle\mathbf{ABC} \cong \triangle\mathbf{WXY}$; in particular, the angle sums of these triangles are equal. On the other hand, the proof of Theorem $7$ implies that the angle sum of $\triangle$**WXY** is equal to $180°$, so the same must be true for $\triangle$**ABC.** ■

This result extends directly to arbitrary triangles in the neutral plane **P.**

**Theorem 10.** *If a rectangle exists in a neutral plane* **P**, *then every triangle in* **P** *has an angle sum equal to* $180°$.

***Proof.*** The idea is simple; we split the given triangle into two right triangles and apply the preceding result. By the results of Unit III, we know that the perpendicular from one vertex of a triangle meets the opposite side in a point between the other two vertices (in particular, we can take the vertex opposite the longest side). Suppose now that the triangle is labeled $\triangle$**ABC** so that the foot **D** of the perpendicular from **A** to **BC** lies on the open segment **(BC).**




We know that **D** lies in the interior of $\angle$**BAC,** and therefore we have

$$|\angle BAD| + |\angle DAC| = |\angle BAC|.$$

By the previous result on angle sums for right triangles, we also have

$$|\angle BAD| + |\angle ADB| = 90^\circ = |\angle DAC| + |\angle ACD|$$

and if we combine all these equations we find that

$$|\angle ABC| + |\angle BCA| + |\angle CAB| = 180^\circ$$

which is the desired conclusion. ■

There is also a converse to the preceding two results.

<u>**Theorem 11.**</u> *If a neutral plane* **P** *contains at least one triangle whose angle sum is equal to* **180°,** *then* **P** *contains a rectangle.*

<u>***Proof.***</u> The idea is to reverse the preceding discussion; we first show that under the given conditions there must be a right triangle whose angle sum is equal to **180°,** and then we use this to show that there is a rectangle.

<u>**FIRST STEP:**</u> If there is a triangle whose angle sum is **180°,** then there is also a right triangle with this property.

Given a triangle whose angle sum is **180°,** as in the previous result we label the vertices **A, B, C** so that the foot of the perpendicular from **A** to **BC** lies on the open segment **(BC).** Reasoning once again as in the proof of Theorem **10** we find

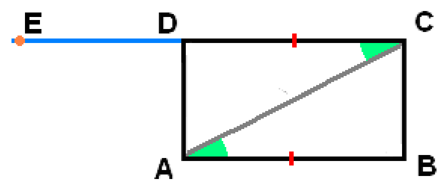
$$\textbf{Angle sum}\,(\triangle ABD) + \textbf{Angle sum}\,(\triangle ADC) =$$

$$\textbf{Angle sum}\,(\triangle ABC) + 180^\circ = 180^\circ + 180^\circ = 360^\circ.$$

Since each of the summands on the left hand side is at most **180°,** it follows that each must be equal to **180°,** (if either were strictly less, then the left side would be less than **360°**). Thus the two right triangles $\triangle$**ABD** and $\triangle$**ADC** have angle sums equal to **180°.**

<u>**SECOND STEP:**</u> If there is a right triangle whose angle sum is **180°,** then there is also a rectangle.

Once again the idea is simple. We shall construct another right triangle with the same hypotenuse to obtain a rectangle. Suppose that $\triangle$**ABC** is the right triangle whose angle sum is equal to **180°,** and that the right angle of this triangle is at **B.**




By the Protractor Postulate there is a unique ray **[CE** such that **(CE** is on the side of **AC** opposite **B** and $|\angle\mathbf{ECA}| = |\angle\mathbf{BAC}|$. Take **D** to be the unique point on **(CE** such that $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{C}, \mathbf{D})$. Then Theorem **7** and **SAS** imply that $\triangle\mathbf{BAC} \cong \triangle\mathbf{DCA}$. In particular, we have $|\angle\mathbf{DAC}| = |\angle\mathbf{BCA}|$ and $|\angle\mathbf{ADC}| = |\angle\mathbf{ABC}|$.

It follows that **AD** and **DC** are perpendicular, so we know there are right angles at **B** and **D.** Furthermore, the Alternate Interior Angle Theorem (more correctly, *<u>the part which is valid in neutral geometry</u>*) implies that the lines **AB** and **CD** are parallel, and likewise the same result and the triangle congruence imply that **AD** and **BC** are parallel. As in the discussion of Lambert quadrilaterals, these conditions imply that **A, B, C, D** form the vertices of a convex quadrilateral. We shall use this to prove that there are also right angles at **A** and **C.**

Since we now know we have a convex quadrilateral, it follows that **A** and **C** lie in the interiors of $\angle\mathbf{BCD}$ and $\angle\mathbf{DAB}$ respectively. Therefore we have

$$|\angle\mathbf{BCD}| = |\angle\mathbf{ACD}| + |\angle\mathbf{ACB}| = |\angle\mathbf{BAC}| + |\angle\mathbf{ACB}| = 90°$$

where the last equation holds because of our assumption about the angle sum of the right triangle $\triangle\mathbf{ABC}$. Thus we know that there also is a right angle at the vertex **C.** But we also have

$$|\angle\mathbf{BAD}| = |\angle\mathbf{BAC}| + |\angle\mathbf{BCA}| = |\angle\mathbf{ACD}| + |\angle\mathbf{BCA}| = 90°$$

where the final equation this time follows because we have shown there is a right angle at **C.** Thus we see that there is also a right angle at **A** and therefore we have a rectangle.■

This brings us to the main result of this section.

**<u>Theorem 12. (All – or – Nothing Theorem)</u>** *In a given neutral plane* $\mathcal{P}$, **EITHER** ***<u>every triangle</u>*** *has an angle sum is equal to* $180°$ **OR ELSE** ***<u>no triangle</u>*** *has an angle sum equal to* $180°$. *In the second case the angle sum of every triangle is strictly less than* $180°$.

***<u>Proof.</u>*** This is mainly a matter of sorting through the preceding results. If one triangle has an angle sum equal to $180°$, then by Theorem **11** a rectangle exists, and in that case Theorem **10** implies that every triangle has angle sum equal to $180°$. Therefore it is impossible to have a neutral plane in which some triangles have angle sums equal to $180°$ but others do not. Finally, by the Saccheri – Legendre Theorem we know that if no triangle has angle sum equal to $180°$ then every triangle must have an angle sum that is strictly less than $180°$.■

> The sum of the three angles of a plane triangle cannot be greater than $180°$… But the situation is quite different in the second part — that the sum of the angles cannot be less than $180°$; this is the critical point, the reef on which all the wrecks occur.
>
> C. F. Gauss, ***Letter to F. (W.) Bolyai***

> When you have eliminated the impossible, whatever remains, however improbable [it may seem], must be the truth.
>
> A. C. Doyle (1859 – 1930), ***Sherlock Holmes – Sign of the Four***

In some respects, the results of this section provide reasons to be optimistic about finding a proof of Euclid's Fifth Postulate in an arbitrary neutral plane. First of all, the results on rectangles and angle sums show that Playfair's Postulate is equivalent to statements that look much weaker (for example, the existence of ***just one rectangle*** or ***just one triangle whose angle sum is*** $180°$). Furthermore, the results suggest that the negation of Playfair's Postulate leads to consequences which seem extremely strange and perhaps even unimaginable. However, as Gauss indicated in his letter, no one was able to overcome the final hurdle and give a complete proof of Euclid's Fifth Postulate from the other axioms for Euclidean geometry. Although the efforts to prove Euclid's Fifth Postulate did not lead to the proof, the best work on the problem provided very extensive, and in some cases nearly complete, information on strange things that would happen if one assumes that the Fifth Postulate is false. We shall examine some of these phenomena in the remaining sections of this unit.

Ultimately these considerations led to a viewpoint expressed in another quotation from Gauss' correspondence:

> The theorems of this geometry appear to be paradoxical and, to the uninitiated, absurd; but calm, steady reflection reveals that they contain nothing at all impossible. (***Letter to Taurinus***, 1824; one should compare this to the Sherlock Holmes quotation given above.)

Before Gauss, some mathematicians (for example, Klügel) had speculated that a proof of the Fifth Postulate might be out of reach. However, Gauss (and to a lesser extent a contemporaries like Schweikart and Taurinus) took things an important step further, concluding that the negation of the Fifth Postulate yields a geometrical system which is very different from Euclidean geometry in some respects but has ***exactly the same degree of logical validity*** (compare also the passage from the letter to Olbers at the beginning of this unit). Working independently of Gauss, J. Bolyai (1802 – 1860) and N. I. Lobachevsky (1792 – 1856) reached the same conclusions as Gauss (each one independently of the other), which Bolyai summarized in a frequently repeated quotation:

> Out of nothing I have created a strange new universe.

Both Bolyai and Lobachevsky took everything one important step further than Gauss by publishing their conclusions, and for this reason they share credit for the first published recognition of hyperbolic geometry as a mathematically legitimate subject.

*Subsequent sections of these notes*

In the remaining sections of this unit we shall consider the mathematical system in which $(i)$ all the axioms for Euclidean geometry except the Parallel Postulate are true, and $(ii)$ the latter is assumed to be false. One goal will be to give an introduction to some basic phenomena in this geometry which are quite different from Euclidean geometry. This will be done in Sections **V.4** and **V.5**; the former will contain proofs at the level of the arguments in this section, but in the latter section we shall concentrate on the results rather than their proofs. Following this, we shall discuss a significant point which the previously mentioned workers anticipated but not quite achieve; namely, **showing beyond all doubt that *it is mathematically impossible to prove the Parallel Postulate from the other axioms*. This was done by *constructing explicit mathematical models***, which are given by data $(\mathbf{P}, \mathcal{L}, d, \mu)$ satisfying both $(i)$ and $(ii)$. Descriptions of the latter appear in Sections **V.6** and **V.7**. In the final Section **V.8** we shall summarize the impact of non – Euclidean geometry as presented in these notes.