

Exercises for Unit III (Basic Euclidean concepts and theorems)

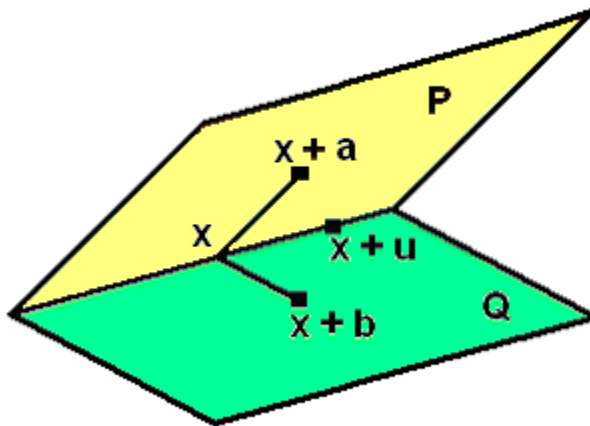
Default assumption:

All points, etc. are assumed to lie in \mathbb{R}^2 or \mathbb{R}^3 .

III.1 : Perpendicular lines and planes

1. Suppose that P , Q and T are three distinct planes, and suppose that they have at least one point in common but do not have a line in common. Prove that they have **exactly** one point in common.

2. Suppose P and Q are two planes which intersect in the line $L = x + U$, where the 1 – dimensional vector subspace U spanned by the unit vector u . Express these planes as translates of two dimensional subspaces, with $P = x + V$ and $Q = x + W$. Let a and b be unit vectors in V and W respectively such that a and b are perpendicular (or normal) to u . Prove that the (cosine of the) angle $\angle(x + a)x(x + b)$ is equal to the (cosine of the) angle between the normals to P and Q ; note that these normals are given by $a \times u$ and $b \times u$. [Hint: Express the dot product of the normals in terms of the dot product of a and b . Apply the formula for $(v \times w) \cdot (y \times z)$ derived in Section I.2.]



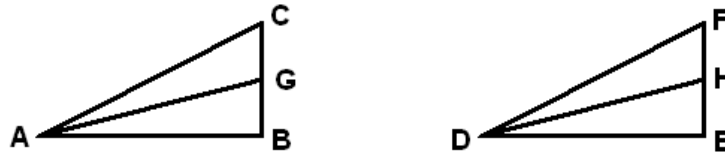
Note. If we let $[L(x + a)]$ denotes the union of L with the set of all points on the same side of P as $x + a$, and we let $[L(x + b)]$ denotes the union of L with the set of all points on the same side of Q as $x + b$, then the union of $[L(x + a)]$ and $[L(x + b)]$ is an example of a **dihedral angle**, and the result of the exercise states that two standard methods for defining the measure of this dihedral angle yield the same value.

3. Let X be a point in the plane P . Prove that there is a pair of perpendicular lines L and M in P which meet at X and that there is no line N in P through X which is perpendicular to both L and M . [Hint: Try using linear algebra.]

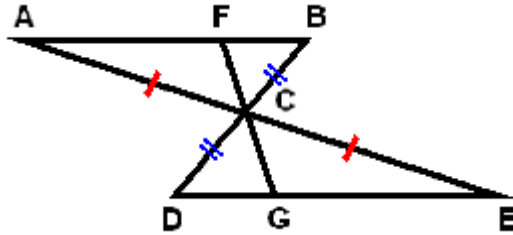
4. Assume the setting of the previous exercise, but also assume that \mathbf{P} is contained in \mathbb{R}^3 . Prove that there is a unique line \mathbf{K} through \mathbf{X} which is perpendicular to both \mathbf{L} and \mathbf{M} .
5. Let \mathbf{L} and \mathbf{M} be lines which intersect at \mathbf{Y} , and for each \mathbf{X} in $\mathbf{L} - \{\mathbf{Y}\}$, let \mathbf{M}_X denote the foot of the unique perpendicular from \mathbf{X} to \mathbf{M} . Prove that for each positive real number a there are exactly two choices of \mathbf{X} for which $d(\mathbf{X}, \mathbf{M}_X) = a$. [*Hint:* Parametrize the line in the form $\mathbf{Y} + t\mathbf{V}$ for some nonzero vector \mathbf{v} , let \mathbf{W} be a nonzero vector such that \mathbf{L} and \mathbf{M} lie in the plane determined by \mathbf{Y} , $\mathbf{Y} + \mathbf{V}$, and $\mathbf{Y} + \mathbf{W}$ with \mathbf{W} perpendicular to \mathbf{V} , and express $d(\mathbf{X}, \mathbf{M}_X)$ in terms of t and the length of \mathbf{W} .]

III.2 : Basic theorems on triangles

1. (Review of topics from Section II.4) Suppose that we are given $\triangle ABC$ and $\triangle DEF$, and let \mathbf{G} and \mathbf{H} denote the midpoints of $[\mathbf{BC}]$ and $[\mathbf{EF}]$ respectively. Prove that $\triangle ABC \cong \triangle DEF$ if and only if $\triangle GAB \cong \triangle HDE$.



2. Suppose that $\triangle ABC$ is an isosceles triangle with $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{A}, \mathbf{C})$, and \mathbf{D} is a point of (\mathbf{BC}) such that $[\mathbf{AD}]$ bisects $\angle \mathbf{BAC}$. Prove that \mathbf{D} is the midpoint of (\mathbf{BC}) and that $|\angle \mathbf{ADB}| = |\angle \mathbf{ADC}| = 90^\circ$.
3. Suppose we are given isosceles $\triangle \mathbf{PRL}$ with $d(\mathbf{R}, \mathbf{P}) = d(\mathbf{L}, \mathbf{P})$. Let \mathbf{S} and \mathbf{T} be points on (\mathbf{RL}) such that $\mathbf{R}^*\mathbf{S}^*\mathbf{T}$, $d(\mathbf{R}, \mathbf{S}) = d(\mathbf{L}, \mathbf{T})$, and $d(\mathbf{P}, \mathbf{S}) = d(\mathbf{P}, \mathbf{T})$. Prove that $\triangle \mathbf{RTP} \cong \triangle \mathbf{LSP}$ and $|\angle \mathbf{PSR}| = |\angle \mathbf{PTL}|$.
4. Suppose we are given two lines \mathbf{AE} and \mathbf{CD} , and suppose that they meet at a point \mathbf{B} which is the midpoint of $[\mathbf{AE}]$ and $[\mathbf{CD}]$. Prove that $\mathbf{AC} \parallel \mathbf{DE}$.
5. Suppose that we are given lines \mathbf{AE} , \mathbf{BD} and \mathbf{FG} which contain a common point \mathbf{C} and also satisfy $\mathbf{A}^*\mathbf{F}^*\mathbf{B}$, $\mathbf{B}^*\mathbf{C}^*\mathbf{D}$, and $\mathbf{D}^*\mathbf{G}^*\mathbf{E}$. Suppose also that $d(\mathbf{A}, \mathbf{C}) = d(\mathbf{E}, \mathbf{C})$ and $d(\mathbf{B}, \mathbf{C}) = d(\mathbf{C}, \mathbf{D})$. Prove that $\triangle \mathbf{ABC} \cong \triangle \mathbf{EDC}$ and $\triangle \mathbf{AFC} \cong \triangle \mathbf{EGC}$. [*Hint:* Part of the proof is to show that the betweenness properties $\mathbf{A}^*\mathbf{C}^*\mathbf{E}$ and $\mathbf{F}^*\mathbf{C}^*\mathbf{G}$, suggested by the drawing below, are true.]

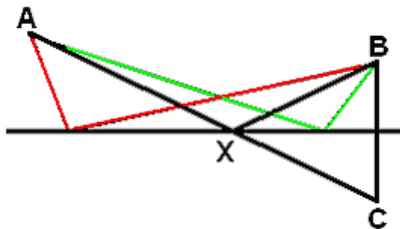


6. Suppose that $\triangle ABC$ is an isosceles triangle with $d(A, B) = d(A, C)$, and let D and E be points of (AB) and (AC) respectively such that $d(A, D) = d(A, E)$. Prove that $BC \parallel DE$.

7. Suppose that we are given $\triangle ABC$, and let D be a point in the interior of $\triangle ABC$ such that $[AD$ bisects $\angle CAB$, $[BD$ bisects $\angle CBA$, and $|\angle ADB| = 130^\circ$. Find the value of $|\angle ACB|$.

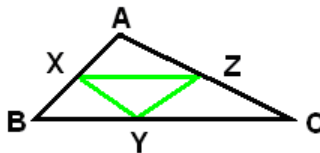
8. Suppose that we are given points A, B, C such that $A*B*C$, and let $DE \neq AC$ such that $D*B*E$, $CE \perp AC$, and $DE \perp AD$. Prove that $|\angle DAB| = |\angle BEC|$.

9. Prove the following result due to Heron of Alexandria: Let P be a plane, let L be a line, let A and B be points on the same side of L in P , and let C be the mirror image of B with respect to L (formally, choose C so that L is the perpendicular bisector of $[BC]$). Define a positive real valued function f on L by $f(X) = d(A, X) + d(X, B)$. Then the minimum value of $f(X)$ occurs when X lies on (AC) .



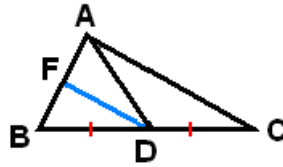
[Hint: Why is $d(X, B) = d(X, C)$, and how is this relevant to the problem?]

10. Given $\triangle ABC$, let X, Y and Z be points on the open segments (AB) , (BC) and (AC) respectively. Prove that the sum of the lengths of the sides of $\triangle ABC$ is greater than the sum of the lengths of the sides of $\triangle XYZ$.

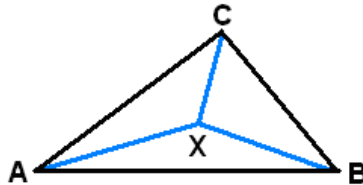


11. Given $\triangle ABC$, let D and E be the midpoints of (BC) and (AC) respectively. Prove that $d(D, E) = \frac{1}{2}d(A, B)$.

12. Given $\triangle ABC$, let D be the midpoint of (BC) . Prove that $d(A, D) < \frac{1}{2}[d(A, B) + d(A, C)]$. [Hint: Let F be the midpoint of (AB) , and apply the previous exercise.]

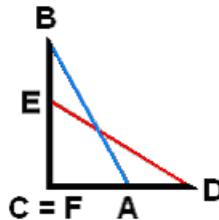


13. Given $\triangle ABC$, let X be a point in the interior of $\triangle ABC$. Prove that $|\angle AXB| + |\angle BXC| + |\angle CXA| = 360^\circ$.



[**Hint:** There is one large triangle in the picture, and it is split into three smaller ones; the angle sum for each triangle is equal to 180° .]

14. Prove the **Sloping Ladder Theorem**: Suppose we are given right triangles $\triangle ABC$ and $\triangle DEF$ with right angles at C and F respectively such that the hypotenuses satisfy $d(A, B) = d(D, E)$ and $d(E, F) < d(B, C)$. Then $d(A, C) < d(D, F)$.



15. In $\triangle ABC$, suppose that $d(A, C) < d(B, C)$ and E is the midpoint of $[AB]$. Is $\angle CEA$ *acute* (measurement less than 90°) or *obtuse* (measurement greater than 90°)? Why?

16. Using the strong triangle inequality for noncollinear triples of points, determine which of the following triples cannot be the set of lengths for the sides of a triangle.

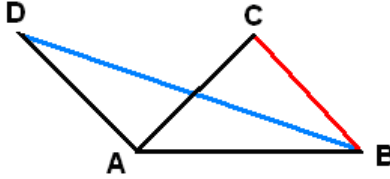
- (a) 1, 2, 3
- (b) 4, 5, 6
- (c) 15, 15, 1
- (d) 5, 1, 8

17. Two sides of a triangle have lengths 10 and 15. Between what two numbers must the length of the third side lie?

18. Let n be a positive integer. Explain why there is a right triangle $\triangle ABC$ with a right angle at C such that (i) the sides all have integral lengths, (ii) $d(A, B) = n + 1$ and $d(A, C) = n$, provided the odd integer $(2n + 1)$ is a perfect square, and conclude that there are infinitely many values of n for which there is a right triangle

$\triangle ABC$ with right angle at C satisfying (i) and (ii). Find all $n < 100$ for which such triangles exist. [**Hint:** Recall that the sum of the first k odd (positive) integers is equal to k^2 .]

19. Prove the **Hinge Theorem**: Given triangles $\triangle ABC$ and $\triangle ABD$ which satisfy $d(A, C) = d(A, D)$, then $d(B, C) < d(B, D)$ if and only if $|\angle CAB| < |\angle DAB|$.



20. Assume that we are given $\triangle ABC$ such that the sides opposite vertices A, B, C have lengths a, b, c and the vertex angles at A, B, C have measures α, β, γ respectively. Then several results of this section show that a, b, c and α, β, γ satisfy certain restrictions. For example, we have [1] the sum of any two lengths is greater than the third, [2] $b = c$ if and only if $\gamma = \beta$ and similarly if the roles of the variables are interchanged, [3] $\alpha + \beta + \gamma = 180^\circ$; [4] $c^2 < a^2 + b^2$ if and only if $\gamma < 90^\circ$. Determine which of these reasons imply that one cannot construct a triangle whose measures are partially given as follows (in any given example more than one reason might be needed):

- (a) $a = 8, b = c = 6, \beta = \gamma = 60^\circ$.
- (b) $a = 6, b = 7, c = 9, \gamma = 93^\circ$.

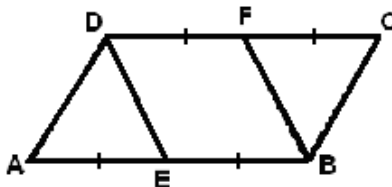
21. In Section II.4 we noted that there is no general Side – Side – Angle (SSA) congruence theorem in geometry. One easy way to construct an explicit counterexample is to start with an isosceles right triangle $\triangle ABC$ with a right angle at the vertex B , take the point D such that $A * C * D$ and $d(B, C) = d(C, D)$, and consider the triangle correspondence $\triangle BDC \leftrightarrow \triangle BDA$, which satisfies the SSA condition. Find the measures of $\angle CBD, \angle ABD, \angle DAB$ and $\angle DCB$. [**Hint:** Why is $\triangle BCD$ isosceles?]

III.3 : Convex polygons

1. Suppose that A, B, C, D form the vertices of a convex quadrilateral, and let P, Q, R, S be the midpoints of $[AB], [BC], [CD]$ and $[DA]$ respectively. Prove that $PQ \parallel RS$ and $QR \parallel PS$. [**Hint:** In each case, the lines are parallel to one of the diagonals of the original convex quadrilateral.]

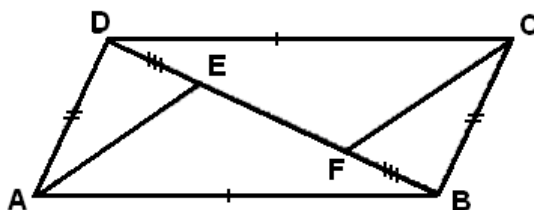
2. Suppose that A, B, C, D form the vertices of a convex quadrilateral, and let P, Q, R, S be the midpoints of $[AB], [BC], [CD]$ and $[DA]$ respectively. Prove that $[PR]$ and $[QS]$ meet at their common midpoint. [**Hint:** Apply the preceding exercise.]

3. Suppose that A, B, C, D form the vertices of a parallelogram, and suppose that E and F are the midpoints of $[AB]$ and $[CD]$ respectively. Prove that E, B, F, D form the vertices of a parallelogram. [*Hint*: There is a simple proof using vectors.]

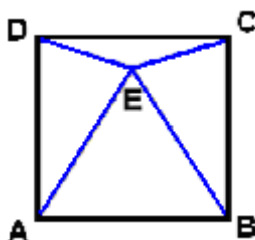


4. Suppose that A, B, C, D form the vertices of a trapezoid, with $AB \parallel CD$, and assume that $d(A, D) = d(C, D)$. Prove that $[AC]$ bisects $\angle DAB$.

5. Suppose that A, B, C, D form the vertices of a parallelogram, and suppose that E and F are points of (BD) such that $B * F * E$ and $d(B, F) = d(D, E)$. Prove that $AE \parallel CF$.



6. A parallelogram is a **rhombus** if its four sides have equal length. Prove that a parallelogram is a rhombus if and only if its diagonals are perpendicular to each other.
7. Suppose that A, B, C, D form the vertices of a square, and let E be a point in the square's interior such that $\triangle ABE$ is an equilateral triangle. Find $|\angle EDC|$ and $|\angle ECD|$.



8. Prove a **converse to Proposition III.3.1**: If A, B, C, D are coplanar points such that no three are collinear, then they form the vertices of a convex quadrilateral if the open diagonal segments (AC) and (BD) have a point in common.

9. Suppose that A, B, C, D are points in \mathbb{R}^3 such that no three are collinear. Prove that they form the vertices of a convex quadrilateral if and only if D lies in the interior of $\angle ABC$ and D and B lie on opposite sides of AC . [*Hint*: Recall that they form the vertices of a convex quadrilateral if and only if the open diagonal segments (AC) and (BD) have a point in common.]

10. Suppose that A, B, C, D are points in \mathbb{R}^2 such that no three are collinear, and write D as an affine combination $D = xA + yB + zC$, where

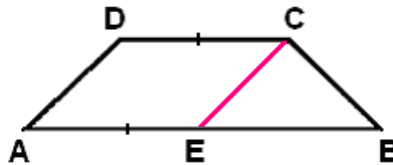
$x + y + z = 1$. Using the preceding exercise, show that A, B, C, D form the vertices of a convex quadrilateral if and only if x and z are positive and y is negative.

11. Suppose that A, B, C, D are points in \mathbb{R}^2 such that no three are collinear, and suppose that $AB \parallel CD$. Prove that A, B, C, D form the vertices of a convex quadrilateral if and only if $C - D$ is a positive multiple of $B - A$ (such a quadrilateral is a parallelogram if $C - D = B - A$ and it is a trapezoid in the other cases).

Standing hypotheses: In Exercises 12 – 16 below, points A, B, C, D in \mathbb{R}^2 form the vertices of a convex quadrilateral such that $AB \parallel CD$. The lengths of $C - D$ and $B - A$ will be denoted by x and y respectively.

12. Prove that the line joining the midpoints of $[AD]$ and $[BC]$ is parallel to AB and CD , and its length is $\frac{1}{2}(x + y)$. Also, prove that the line joining the midpoints of the diagonals $[AC]$ and $[BD]$ is parallel to AB and CD .

13. Suppose that $x < y$, and let E be the unique point on (AB) such that $d(A, E) = x$. Prove that E lies on (AB) and $AD \parallel CE$ (hence A, E, C, D form the vertices of a parallelogram).

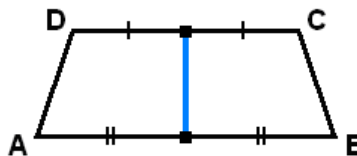


14. Suppose again that $x < y$, and let E be as in the preceding exercise. Prove that the following are equivalent:

- (1) $d(A, D) = d(B, C)$
- (2) $|\angle DAB| = |\angle CBA|$
- (3) $|\angle ADC| = |\angle BCD|$

A trapezoid satisfying one (and hence all) of these conditions is called an **isosceles trapezoid**.

15. Suppose that A, B, C, D as above are the vertices of an isosceles trapezoid. Prove that the line joining the midpoints of $[AB]$ and $[CD]$ is the perpendicular bisector of these segments.

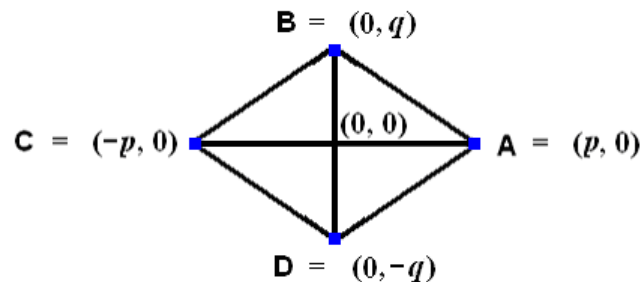


16. Suppose we are given an isosceles trapezoid as in the preceding exercise such that $A = (-\frac{1}{2}y, 0)$, $B = (\frac{1}{2}y, 0)$, $C = (\frac{1}{2}x, h)$, and $D = (-\frac{1}{2}x, h)$, where $h > 0$. Prove that the open diagonal segments (AC) and (BD) meet at a point $(0, k)$ on the y -axis, and express $k/(h - k)$ in terms of x and y .

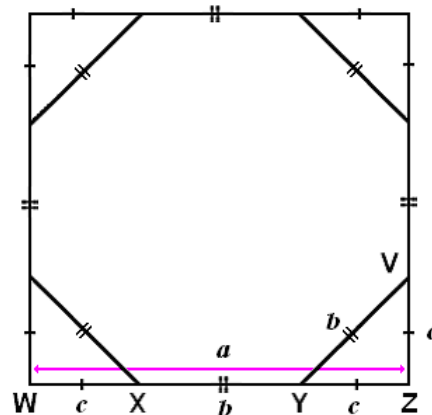
17. Suppose that we are given $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ in \mathbb{R}^2 whose coordinates are given by $\mathbf{A} = (p, 0)$, $\mathbf{B} = (0, q)$, $\mathbf{C} = (-p, 0)$ and $\mathbf{D} = (0, -q)$, where $p, q > 0$.

(a) Prove that $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ form the vertices of a rhombus. [*Hint:* First of all, show that $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{B}, \mathbf{C}) = d(\mathbf{C}, \mathbf{D}) = d(\mathbf{D}, \mathbf{A})$. Next, note that $\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{C}$ and use this to show that $\mathbf{AB} \parallel \mathbf{CD}$. Finally, modify the preceding step to show that $\mathbf{AD} \parallel \mathbf{BC}$.]

(b) Prove that the distance between the parallel lines \mathbf{AB} and \mathbf{CD} is equal to the distance between the parallel lines \mathbf{AD} and \mathbf{BC} . [*Hint:* Let \mathbf{T} be the orthogonal linear transformation defined by $\mathbf{T}(x, y) = (x, -y)$, and view \mathbf{T} as an isometry of \mathbb{R}^2 . What are the images of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ under \mathbf{T} ? What are the images of the lines $\mathbf{AB}, \mathbf{AD}, \mathbf{BC}$ and \mathbf{CD} under \mathbf{T} ? Using these conclusions, prove that if $\mathbf{F} \in \mathbf{AB}$ and $\mathbf{G} \in \mathbf{CD}$ are such that the line \mathbf{FG} is perpendicular to both \mathbf{AB} and \mathbf{CD} , then $\mathbf{T}(\mathbf{F}) \in \mathbf{AD}$ and $\mathbf{T}(\mathbf{G}) \in \mathbf{BC}$ are such that the line $\mathbf{T}(\mathbf{F})\mathbf{T}(\mathbf{G})$ is perpendicular to both \mathbf{AD} and \mathbf{BC} . Why will the result follow from this?]

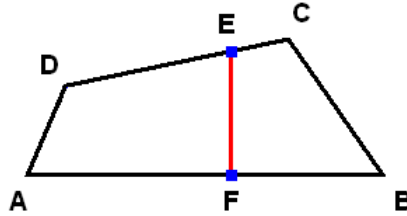


18. Given a square whose sides all have length a , it is possible to obtain a regular octagon by cutting away four isosceles right triangles at the edges as suggested by the figure below. Suppose that b is the length of the sides of the regular octagon constructed in this fashion. Express the value of b in terms of a . [*Hint:* Let c be equal to the lengths of the legs of the isosceles right triangles that are removed to form the octagon. Find two equations relating a, b and c .]



19. Suppose that we are given four points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ in \mathbb{R}^2 which form the vertices of a convex quadrilateral (in the given order). Let \mathbf{E} be a point on (\mathbf{CD}) , and let \mathbf{F} be

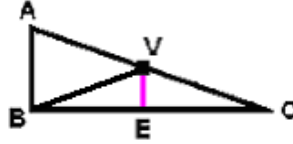
the foot of the perpendicular from E to AB . Prove that if F lies on (AB) , then (EF) is contained in the interior of the convex quadrilateral $ABCD$.



20. Prove an analog of Pasch's Theorem for convex quadrilaterals: Suppose that A, B, C, D in \mathbb{R}^2 form the vertices of a convex quadrilateral and L is a line in \mathbb{R}^2 which contains exactly one point of (AB) . Prove that either L contains one of the vertices C, D or else it contains a point from one of $(BC), (CD)$ or (AD) .
21. Suppose we are given four points A_1, A_2, A_3, A_4 in \mathbb{R}^2 such that no three are collinear, let $A_5 = A_1$, and let $L_i = A_i A_{i+1}$, so that exactly two of the given four points are on the line L_i and the other two are not. Prove that for at least one choice of i the "remaining two" points of $\{A_1, A_2, A_3, A_4\}$ both lie on the same side of L_i .
22. A convex quadrilateral $ABCD$ is said to be a **convex kite** (or **deltoid**) if $d(A, B) = d(A, D)$ and $d(C, B) = d(C, D)$. Prove that the line of the diagonal AC in such a quadrilateral is the perpendicular bisector of the diagonal segment $[BD]$, and also prove that $|\angle ABC| = |\angle ADC|$. Also, prove that $|\angle BCD| = |\angle DAB|$ if and only if convex quadrilateral $ABCD$ is a rhombus.

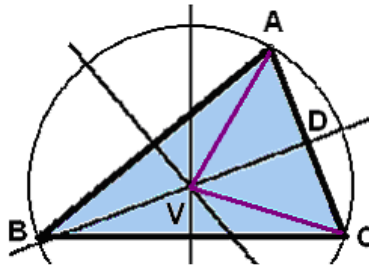
III.4 : Concurrence theorems

- Let $\triangle ABC$ be a triangle in \mathbb{R}^2 . Define a real valued function g on \mathbb{R}^2 by $g(X) = d(X, A)^2 + d(X, B)^2 + d(X, C)^2$. Prove that $g(X)$ takes a minimum value when X is the centroid of $\triangle ABC$.
- Suppose that the circumcenter V of $\triangle ABC$ lies in the interior of that triangle. Prove that all three vertex angles of that triangle are acute (*i.e.*, measure less than 90°). [**Hint:** Consider the three triangles $\triangle VBC, \triangle VAB, \triangle VAC$. Explain why they are isosceles, and show that $|\angle VBC| + |\angle VAB| + |\angle VAC| = 90^\circ$.]
- Suppose we have a right triangle with a right angle at B , and let V be the midpoint of $[AC]$. Explain why $d(V, B) = d(A, C)$, so that V is the circumcenter of the triangle. [**Hint:** Using the final result in Section I.4 of the notes, show that the foot of the perpendicular from V to BC is the midpoint E of the segment $[BC]$. Why does this imply that $\triangle VBC$ isosceles?]



4. Suppose that we have a triangle $\triangle ABC$ such that the circumcenter V lies in the interior of the triangle, and let R be the radius of that circle. Let $|\angle BAC| = \beta$. Prove the following strong version of the Law of Sines:

$$\frac{\sin \beta}{b} = \frac{1}{2 \cdot R}$$



[**Hint:** Let D be the midpoint of $[AC]$, and find $d(V, A)$ and $|\angle VAC|$ in terms of b , β and R . What does this imply about $d(A, D) = \frac{1}{2}b$? You might want to use some of the conclusions obtained in the solution to Exercise 2.]

Note. The solution to Exercise 3 implies similar results for triangles with one right angle, and in fact the same conclusion holds if one of the angles in the triangle is obtuse (the argument is similar but slightly more complicated).

5. The following instructions were found on an old map:

Start from the right angle crossing of King's Road and Queen's Road. Proceed due north on King's Road and find a large pine tree and then a maple tree. Return to the crossroads. Due west on Queen's Road there is an elm, and due east on Queen's Road there is a spruce. One magical point is the intersection of the elm – pine line with the maple – spruce line. The other magical point is the intersection of the spruce – pine line with the elm – maple line. The treasure lies where the line through the magical points meets Queen's Road.

A search party found the elm 4 miles from the crossing, the spruce 2 miles from the crossing, and the pine 3 miles from the crossing, but they found no trace of the maple. Nevertheless, they were able to locate the treasure from the instructions. Show how they were able to do this. [**Hints:** The treasure was eight miles east of the crossing. Probably the best way to do this problem is to set up Cartesian coordinates with King's Road and Queen's Road as the coordinate axes.]

6. Find the circumcenter of the triangle in the coordinate plane with vertices $(1, 1)$, $(5, 5)$ and $(4, 0)$.

7. Find the orthocenter of the triangle in the coordinate plane with vertices $(\pm 1, 0)$ and $(0, 2)$. [**Hint:** The line **L** joining $(1, 0)$ and $(0, 2)$ has equation $y + 2x = 2$. Find the equation of the line **M** which is perpendicular to **L** and passes through $(1, 0)$. Explain why the orthocenter is the point where **M** meets the $y -$ axis.]

Note. For both Exercises 6 and 7, the numerical answers for the coordinates are expressible in relatively neat terms. The same applies to Exercise 9.

8. Find the incenter of $\triangle ABC$ if $A = (1, 0)$, $B = (0, 0)$, and $C = (0, \sqrt{3})$; the conditions of the problem imply that there is a right angle at **B** and a 60° angle at **C**. [**Hint:** The bisector for the angle at **B** has equation $y = x$. Find the equation of the line which bisects the angle at **A**.]

9. Find the circumcenter of the triangle in \mathbb{R}^2 with vertices $(0, 0)$, $(3, 4)$, and $(6, 0)$, and determine the circumradius of this triangle.

10. Given a $120^\circ - 30^\circ - 30^\circ$ isosceles triangle $\triangle ABC$, determine whether each of the circumcenter and orthocenter lies inside the triangle, on one of the vertices, on one of the sides between two vertices, or outside the triangle. You may assume that the legs of the isosceles triangle are **[AB]** and **[AC]**, where **A** lies on the $y -$ axis and **[BC]** is contained in the $x -$ axis.

11. Suppose that we are given three noncollinear points a, b, c in \mathbb{R}^2 or \mathbb{R}^3 . Prove that the angle bisector for $\angle abc$ is the ray **[bx]**, where $x = b + y$ and y is given by

$$\frac{1}{2} \left(\frac{a - b}{\|a - b\|} + \frac{c - b}{\|c - b\|} \right).$$

[**Hint:** Let u and v be the unit vectors in the displayed expression, and let $p = b + u$ and $q = b + v$, so that $\angle abc = \angle pbq$. Then x is the midpoint of **[pq]**; note that $\triangle qbp$ is isosceles.]

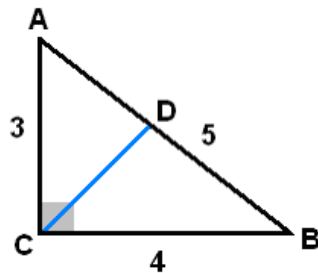
12. Apply the result of the previous exercise to the angle in \mathbb{R}^2 with $a = (3, 4)$, $b = (0, 0)$ and $c = (2, 1)$ to find a point x on the bisector of $\angle abc$ other than b , and find the slope of the line **bx** (which goes through the origin).

III.5 : Similarity

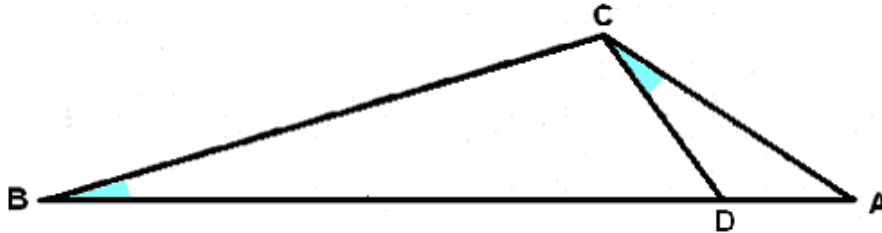
1. Prove that an affine transformation which preserves perpendicularity must be a similarity transformation.
2. Let **T** be a similarity transformation of \mathbb{R}^n with a ratio of similitude k which is **not** equal to 1. Prove there is a unique point z such that $T(z) = z$. [**Hint:** Write

$T(\mathbf{z}) = k\mathbf{A}\mathbf{z} + \mathbf{b}$, where \mathbf{A} is given by an orthogonal matrix. Then the conclusion is equivalent to saying that there is a unique \mathbf{z} such that $(k\mathbf{A} - \mathbf{I})\mathbf{z} = \mathbf{b}$. By linear algebra the latter happens if and only if there is no nonzero vector \mathbf{v} such that $k\mathbf{A}\mathbf{v} = \mathbf{v}$. Assume to the contrary that such a vector exists, and, using the orthogonality of \mathbf{A} , explain why the length of the vector on the left side is equal to $k|\mathbf{v}|$, and note that the length of the vector on the right side is just $|\mathbf{v}|$. Why does this yield a contradiction?]

3. Let $\triangle ABC$ be a 3-4-5 right triangle with a right angle at C such that $d(A, C) = 3$, $d(B, C) = 4$, and $d(A, B) = 5$. Let D be the point on (BC) such that $[CD$ bisects $\angle ACB$. Compute the distances $d(A, D)$ and $d(D, B)$.

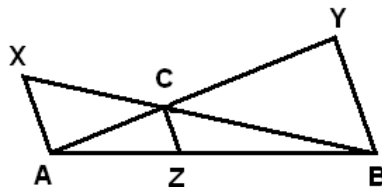


4. Suppose we are given $\triangle ABC$, and let $D \in (AB)$ be such that $|\angle DCA| = |\angle ABC|$. Prove that $d(A, C)$ is the mean proportional between $d(A, B)$ and $d(A, D)$.



5. Let $\triangle ABC$ be given, and let Z be a point on (AB) . Let X and Y be points on the same side of AB as C such that AX, CZ and BY are all parallel to each other, and also assume that $B \cdot C \cdot X$ and $A \cdot C \cdot Y$. Prove that

$$\frac{1}{d(C, Z)} = \frac{1}{d(A, X)} + \frac{1}{d(B, Y)}.$$



6. Suppose that we are given positive real numbers a_1, \dots, a_n and b_1, \dots, b_n such that

$$\frac{a_i}{b_i} = \frac{a_1}{b_1}$$

for all i . Prove that

$$\frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} = \frac{a_1}{b_1}.$$

7. (i) Suppose that $\triangle ABC$ and $\triangle DEF$ satisfy $\triangle ABC \sim \triangle DEF$ with $d(B, C) \leq d(A, C) \leq d(A, B)$. Prove that $d(E, F) \leq d(D, F) \leq d(D, E)$.
- (ii) Suppose that we are given $\triangle ABC$ with $d(B, C) \leq d(A, C) \leq d(A, B)$ and that (A', B', C') is a rearrangement of (A, B, C) such that $\triangle ABC \sim_k \triangle A'B'C'$. Prove that $k = 1$. [**Hint:** Split the proof into cases depending upon whether $\triangle ABC$ is equilateral, isosceles with the base shorter than the legs, isosceles with the legs shorter than the base, or *scalene*; *i.e.*, no two sides have equal length.]
8. Suppose that we are given $\triangle ABC \sim \triangle DEF$, and let G and H be the midpoints of $[BC]$ and $[EF]$ respectively. Prove that $\triangle ABG \sim \triangle DEH$.
9. Suppose we are given the right triangle $\triangle ABC$ such that $d(A, B) = 13$, $d(A, C) = 5$ and $d(B, C) = 12$, so that C is the right angle vertex. If $X \in (AB)$ is such that $[CX]$ bisects $\angle ACB$, find $d(A, X)$.
10. Suppose we are given an isosceles triangle $\triangle ABC$ in the coordinate plane \mathbb{R}^2 whose vertices are given by $A = (0, h)$, $B = (-x, 0)$, and $C = (x, 0)$. Then the incenter J (where the angle bisectors meet) lies on the y -axis. Find its y -coordinate. [**Hint:** What does the Angle Bisector Theorem imply?]

III.6 : Circles and classical constructions

1. Let Γ be a circle with center Q , let $[AB]$ and $[CD]$ be chords of Γ (so that the endpoints lie on the circle), and let G and H be the midpoints of $[AB]$ and $[CD]$. Prove that $d(Q, G) = d(Q, H)$ if and only if $d(A, B) = d(C, D)$, and $d(Q, G) < d(Q, H)$ if and only if $d(A, B) > d(C, D)$.
2. Let Γ be a circle with center Q , and let L be a line containing a point X on Γ . Prove that X is the only common point of Γ and L if and only if QX is perpendicular to L . (These are the usual synthetic descriptions for the *tangent line* to Γ at X .) [**Hint:** If L also meets Γ at another point Y , explain why $\angle QXY$ is acute.]
3. Let Γ be a circle with center Q , let X be a point in the exterior of Γ , and let A and B be two points of Γ which lie on opposite sides of QX such that XA and XB are tangent to Γ in the sense of Exercise 2. Prove that $d(X, A) = d(X, B)$.

4. Let Γ be a circle in the plane, let A be a point in the interior of Γ , and let X be a point different from A . Prove that the ray $[AX$ meets Γ in exactly one point. [*Hint:* By the line – circle theorem, the line AX meets Γ in two points B and C . Why do these points lie on opposite rays?]

5. (**SsA Congruence Theorem for Triangles**) Suppose we have $\triangle ABC$ and $\triangle DEF$ such that $|\angle CAB| = |\angle FDE|$ and $d(B, C) = d(E, F) > d(A, B) = d(D, E)$. Prove that $\triangle ABC \cong \triangle DEF$ by supplying reasons for the steps listed below:

- (1) There is a point $G \in (AC$ such that $d(A, G) = d(E, F)$.
- (2) $\triangle GAB \cong \triangle FDE$.
- (3) $d(B, G) = d(E, F) = d(B, C)$.
- (4) G lies on the circle Γ with center B and radius $d(B, C)$.
- (5) A lies in the interior of Γ .
- (6) $(AG$ meets Γ in exactly one point.
- (7) C lies on $(AG$ and Γ .
- (8) $C = G$.
- (9) $\triangle ABC = \triangle ABG$, and $\triangle ABC \cong \triangle DEF$.

6. Let Γ be a circle whose center is Q , and let A be a point in the same plane that is not on Γ and not equal to Q . Prove that the distance from A to a point X on Γ is minimized for a point Y which also lies on the open ray $(QA$. [*Hint:* There are two separate cases depending upon whether A is in the interior or exterior of the circle. In the first case the point Y satisfies $Q*A*Y$, and in the second case it satisfies $Q*Y*A$. Show first that if W is the other point on $\Gamma \cap QA$, then the distance is not minimized at W ; this leaves us with the cases where X does not lie on QA . The “larger angle is opposite the greater side” theorem is useful in the two separate cases when X does not lie on QA .]

7. Let Γ_1 and Γ_2 be concentric circles in the same plane, let Q be their center, and suppose that the radius p of Γ_1 is less than the radius q of Γ_2 . What is the set of all points X such that the shortest distance from X to Γ_1 equals the shortest distance from X to Γ_2 ? Give a proof that your assertion is correct.

8. Prove the assertion in the notes about finding a triangle with given **SAS** data: Specifically, if we are given positive real numbers b and c , and α is a real number between 0 and 180 , then there is a triangle $\triangle ABC$ such that $d(A, B) = c$, $d(A, C) = b$, and $|\angle CAB| = \alpha^\circ$.

9. Prove that a line cannot contain three distinct points of a circle, or equivalently that no three points of a circle are collinear. [*Hints:* Let L be the line, and let Q be the center of the circle, and suppose that L contains three points of the circle Γ with

center Q and radius r . There are two cases depending upon whether or not $Q \in L$. Use the Ruler Axiom to prove the result in the first case. In the second case, if the circle contains three points of the circle, explain why we can label them X, Y, Z such that $X*Y*Z$. Show that $|\angle QXY| = |\angle QYX| = |\angle QZY| = |\angle QYZ|$ using the Isosceles Triangle Theorem. On the other hand, why do we also know that $|\angle QYX| + |\angle QXY| = 180^\circ$, and why does this yield a contradiction?]

10. Let A, B, C be three distinct noncollinear points. Prove that there is a unique circle containing them.

11. Let A and B be distinct points, let D be the midpoint of $[AB]$, and let Ω be the set of all points X such that $X = A$, $X = B$, or X does not lie on AB and $AX \perp XB$. Prove that Ω is the circle with center D and radius $\frac{1}{2}d(A, B)$.

12. Let L and M be the coordinate axes in \mathbb{R}^2 , and let S be the set of all points Z such that $d(Z, L) + d(Z, M) = d(Z, Q)^2$, where $d(Z, K)$ is distance from a point Z to the foot of the perpendicular to the line K containing Z and Q denotes the origin. Prove that S is the union of four circular arcs; describe each arc in terms of its endpoints, the centers of the circles, and whether it is a minor arc, semicircle or major arc. [*Hint:* Look first at the set of points in S which also lie in the closed first quadrant of points whose coordinates are both nonnegative.]

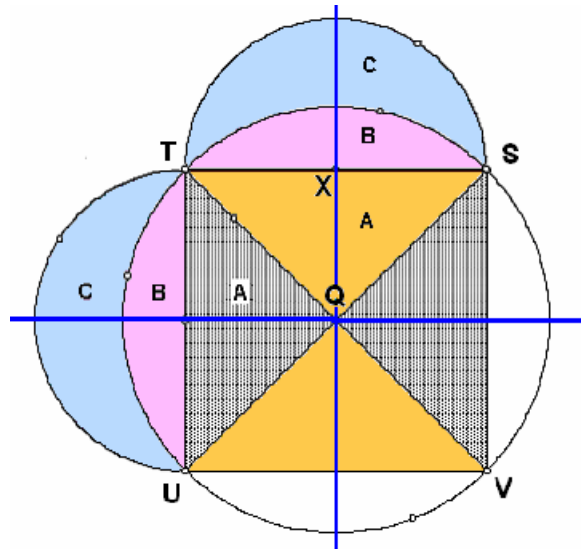
13. Let Γ be a circle with center Q , let A be a point on that circle, and let Ω be the set of all points X such that either $X = A$ or else is the midpoint of the segment $[AB]$ for some $B \in \Gamma$. Prove that Ω is a circle whose center is the midpoint of $[AQ]$. [*Hint:* First consider the special case where Q is the origin and A is the point with coordinates $(a, 0)$; use coordinates to prove the result in this case. One can then modify the argument to work in the general case by taking U to be the unit vector in the direction of $A - Q$ and defining V to be a unit vector in a perpendicular direction.]

14. Let Γ be a circle with center Q , let A, B, C, D be four points on Γ such that Q does not lie on AB or CD , and let E and F be the feet of the perpendiculars from Q to AB and CD respectively. Prove that $d(A, B) = d(C, D)$ if and only if $d(Q, E) = d(Q, F)$. [This is often stated in the form, two chords of a circle have equal length if and only if the distances from the center of the circle to these chords are equal.]

15. Suppose that A, B, C, D are the vertices of a convex kite as defined in Exercise III.3.22. Prove that there is a circle Γ such that all four sides of the quadrilateral $ABCD$ are tangent lines to Γ . [*Hint:* By the definition of a convex kite we have $d(A, B) = d(A, D)$ and $d(C, B) = d(C, D)$. Explain why the diagonal $[AC]$ is contained in the bisectors for both $|\angle DAB|$ and $|\angle BCD|$, explain why the bisector of $\angle ABC$ meets the open diagonal (AC) in some point Q , and explain why $[DQ]$ bisects $\angle ADC$. Finally, explain why the point Q is equidistant from all four sides of $ABCD$, and use this to find the circle Γ .]

III.7 : Areas and volumes

1. Prove that the area of the region bounded by a rhombus is equal to half the product of the lengths of its diagonals.
2. Using Heron's Formula, derive a formula for the area of the region bounded by an equilateral triangle whose sides all have length equal to a .
3. Is there a formula for the area of the region bounded by a convex quadrilateral in terms of the lengths of the four sides (and nothing else)? Give reasons for your answer. **[Footnote:** Compare this with the formula of Brahmagupta, which is stated in the notes and is valid if the vertices all lie on a circle.]
4. Suppose that the radius of the circle inscribed in $\triangle ABC$ is equal to q . Using Heron's Formula, prove that q is equal to $\sqrt{(s-a)(s-b)(s-c)/s}$. **[Hint:** Look at the drawing for Theorem III.4.8, and explain why this figure leads to a formula for the area of the triangle in terms of q and s .]
5. In the drawing below, the blue lines are the axes in the coordinate plane, the points S, T, U and V have coordinates $(1, 1), (-1, 1), (-1, -1)$ and $(1, -1)$ respectively, and the large circle containing them has equation $x^2 + y^2 = 2$. The point Q is the center of this circle, the point X is $(0, 1)$, the smaller semicircles have radius 1 , and the numbers A, B, C denote the areas of the regions bounded by the appropriate curves. Using the standard formula $\text{AREA}(\Gamma) = \pi r^2$ for a solid region bounded by a circle Γ of radius r , show that $A = C$ and use this to evaluate C explicitly as a radical expression involving positive integers. **An informal argument will be acceptable.** Here is a drawing which depicts the data of the problem:



[Hint: Show that the area $B + C$ of the smaller semicircular regions is half the area of the semicircular region bounded by $[US]$ and the semicircle containing U, S and T .]

Notes: This relationship was discovered by Hippocrates of Chios (470 – 410 B. C. E.), who was not the same person as the celebrated physician Hippocrates of Kos (460 – 377 B. C. E.). Further information on problems of this sort is summarized following the solution to this exercise (see the **solutions** file for this section), and there also is a detailed discussion of the topic at a fairly elementary level in Chapter **10** of the following book:

T. Dantzig, *Mathematics in Ancient Greece* (Reprint of the **1955** book, *The Bequest of the Greeks*). Dover, New York, **2006**.

The original edition of this book is also available at the following online site:

<http://ia310936.us.archive.org/1/items/bequestofthegree032880mbp/bequestofthegree032880mbp.pdf>