SOLUTIONS TO EXERCISES ON AFFINE EQUIVALENCE

Here are the solutions to the additional exercises in `affexercises.pdf`.

T 1. No points of K lie on L, so either all points lie on one side of L or else L contains points $\mathbf{x}$ and $\mathbf{y}$ on opposite sides of L. In the latter case, there is a point $\mathbf{z} \in (\mathbf{xy}) \cap L$. By convexity we also have $\mathbf{z} \in K$, so that L and K have a point in common, contradicting our original assumption. The source of this contradiction is the assumption that K contains points on both sides of L, so this must be impossible.■

T 2. By Corollary II.8.4 we know that the affine transformation F must send the ray $[\mathbf{ba}$ to $[F(\mathbf{b})F(\mathbf{a})$, and likewise F sends $[\mathbf{bc}$ to $[F(\mathbf{b})F(\mathbf{c})$. As noted in the hint, F also maps a union $X \cup Y$ to $F[X] \cup F[Y]$; since an angle is the union of two noncollinear rays with the same endpoint, these combine to show that F maps $\angle\mathbf{abc}$ to $\angle F(\mathbf{a})F(\mathbf{b})F(\mathbf{c})$.■

T 3. We know that $\mathbb{R}^2$ is the union of the three pairwise disjoint subsets $\angle\mathbf{abc}$, Interior $(\angle\mathbf{abc})$ and Exterior $(\angle\mathbf{abc})$. Since F is a 1–1 correspondence from $\mathbb{R}^2$ to itself, it follows that the images of these three subsets are pairwise disjoint subsets whose unions are all of $\mathbb{R}^2$. By the preceding exercise we know F maps $\angle\mathbf{abc}$ to $\angle F(\mathbf{a})F(\mathbf{b})F(\mathbf{c})$, and by Theorem 12 in `affine-convex.pdf` we know that F also maps Interior $(\angle\mathbf{abc})$ to the interior of $\angle F(\mathbf{a})F(\mathbf{b})F(\mathbf{c})$. Therefore F must also send

$$\text{Exterior}\,(\angle\mathbf{abc}) \quad = \quad \mathbb{R}^2 \;-\; \big(\,\angle\mathbf{abc} \;\cup\; \text{Interior}\,(\angle\mathbf{abc})\,\big)$$

to the complement of $\angle F(\mathbf{a})F(\mathbf{b})F(\mathbf{c}) \cup$ Interior (same). Since the latter set is the exterior of $\angle F(\mathbf{a})F(\mathbf{b})F(\mathbf{c})$, it follows that it must be the image of Exterior $(\angle\mathbf{abc})$ under F.■

T 4. (i) Since $A * C * B$ is true it follows that A and C lie on the same side of M, and likewise it follows that B and C lie on the same side of L. Hence the strip is nonempty because there is at least one point $A \in L$ and at least one point $B \in M$. By definition the strip is the intersection of two convex sets; since an intersection of convex sets is also convex, the strip itself is convex.■

(ii) As in the preceding result let $A \in L$ and $B \in M$. By Theorem 11 in `affine-convex.pdf` it follows that F sends the side of L containing B to the side of $F[L]$ containing $F(B)$ and it also sends the side of M containing A to the side of $F[M]$ containing $F(A)$. By definition the intersection of the two images is the strip between $F[L]$ and $F[M]$, and therefore F maps one parallel strip to the other as asserted in the exercise.■

T 5. If $\mathbf{p} \in K_1$, then either $\mathbf{p} \in H_1$ or $\mathbf{p} \in H_2$ because $\mathbf{p}$ lies in $\mathbb{R}^2 - L$.

Case 1. Assume $\mathbf{p} \in H_1$. Then the solution to Exercise T1 implies that $K_1 \subset H_1$. Similarly, since $\mathbf{p} \in K_1 \subset H_1$ we must also have $H_1 \subset K_1$, so that $H_1 = K_1$. Now assume that $\mathbf{q} \in K_2$, so that either $\mathbf{q} \in H_1$ or $\mathbf{q} \in H_2$. If $\mathbf{q} \in H_1$, then the same considerations as before show that $K_2 \subset H_1$ and hence $\mathbb{R}^2 - L \subset H_1$, which we know is false. Therefore we must have $\mathbf{q} \in H_2$, and by the previous reasoning this implies that $K_2 \subset H_2$, which in turn implies that $K_2 = H_2$.

Case 2. Assume $\mathbf{p} \in H_2$. If we switch the roles of H_1 and H_2 in the preceding paragraph, we obtain the conclusion that $K_1 = H_2$ and $K_2 = H_1$.■