SOLUTIONS TO MORE ADDITIONAL EXERCISES FOR II.1

Here are the solutions to the additional exercises **J1.–J4.**

J1. We need to verify each of the axioms.

(I–1) For the given system, this axiom translates into the statement, "Given two distinct points, there is a unique two point subset containing them." Since this statement is automatically true for any set, it is true for the system which we have defined.■

(I–2) Every line contains at exactly two points in this case because lines were defined to be two point subsets.■

(I–3) Since lines contain exactly two points, no three point set is collinear. As in the discussion of the first axiom, the third axiom translates into the statement, "Given three distinct points, there is a unique three point subset containing them." Since this statement is automatically true for any set, it is true for the system which we have defined.■

(I–4) By construction every plane contains exactly three points.■

(I–5) In this case the axioms translate into the following: "If P is a three point subset and $A, B \in P$, then the two point subset $\{A, B\}$ is contained in P. Once again this is true for any set, so it is true in our system.■

(I–6) Suppose that P and Q are distinct three point subsets of $\mathbb{S}$. Then there is some point in P which is not in Q or *vice versa*. In either case, this means that $P \cup Q$ contains at least four points. Since $P \cup Q$ is a subset of $\mathbb{S}$ and the latter contains exactly four points, it follows that $P \cup Q = \mathbb{S}$. We can now use the standard counting formula for fintite sets; namely, if $|X|$ denotes the number of elements in a finite set X, then for any two finite sets A and B we have

$$|A \cup B| \;=\; |A| + |B| - |A \cap B| \ .$$

In our particular situation this yields

$$4 \;=\; |\mathbb{S}| \;=\; 3 + 3 - |P \cap Q|$$

which means that $|P \cap Q| = 2$ and hence the two planes do have a second point in common.■

Conclusion. The preceding system satisfies the incidence axioms, but its points and lines are finite. If the incidence axioms implied that points and/or lines were infinite, this would not be possible.■

J2. By definition, there is some plane P_0 containg the lines L and M; we need to prove it is unique. Since $L \neq M$ we know that there is some point $Y \in L$ such that $Y \notin M$. If Q is an arbitrary plane containing both lines, then Q must contain L and Y, and therefore by Proposition II.1.2 there is only one plane containing L and Y. Since P_0 is such a plane, we must have $Q = P_0$.■

J3. If P and Q are distinct planes containing the two distinct points A and B, then by **(I–5)** we know that $P \cap Q$ contains the line AB. If the intersection contained yet another point Y not on this line, then Proposition II.2.1 would imply that $P = Q$, so the intersection must be precisely AB. Therefore, if C is a third point in $P \cap Q$, then we must have $C \in AB$, so that the original triple of points is collinear.■

J4. Suppose that P_1, P_2 and P_3 are distinct planes whose intersection contains at least two points A and B. Then by **(I–5)** we know that each of the three planes contains the line AB. On the other hand, the reasoning of the previous exercise shows that $P_1 \cap P_2$ must be exactly AB, and therefore we have

$$AB \quad \subset \quad P_1 \cap P_2 \cap P_3 \quad \subset \quad P_1 \cap P_2 \quad = AB$$

which implies that $P_1 \cap P_2 \cap P_3$ must be equal to AB.

Here is one set of examples:

The three planes $z = 0$, $z = 1$ and $z = 2$ in $\mathbb{R}^3$ have no points in common.

The intersection of the $xy-$, $yz-$ and $xz-$ planes in $\mathbb{R}^3$ is the origin.

The intersection of the $yz-$plane, the $xz-$plane and the plane $y = x$ (in space) is equal to the $z-$axis.

Obviously there are many further examples for each possibility.■