**Note on coplanar lines in $\mathbb{R}^n$**

In Section II.5 of the course notes there is a proof that if $V$ is a 1-dimensional vector subspace of $\mathbb{R}^n$ and $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ are vectors and the lines $\mathbf{x} + V$ and $\mathbf{y} + V$ are not the same, then the lines $\mathbf{x}+V$ and $\mathbf{y}+V$ are parallel; in other words, they are coplanar and disjoint. Conversely, experience indicates that if $\mathbf{x} + U$ and $\mathbf{y} + V$ are coplanar lines such that $U \neq V$, then $\mathbf{x} + U$ and $\mathbf{y} + V$ have a point in common. *Our purpose here is to give a proof of this fact using vectors.*

The first steps in proving this assertion are to make some simple but necessary observations.

Suppose we are given two lines as above. Then they are unequal because otherwise we have

$$0 \;\in\; (\mathbf{y} - \mathbf{x}) + V \;=\; (\mathbf{y} - \mathbf{y}) + U \;=\; U$$

so that $(\mathbf{y} - \mathbf{x}) + V = V$ and hence $V = U$, contradicting our assumptions on these vector subspaces. Since both subspaces are 1-dimensional they are spanned by single nonzero vectors $\mathbf{u} \in U$ and $\mathbf{v} \in V$.

Assume now that the two lines are coplanar, so that there is some plane $P = \mathbf{z} + W$ containing both of them; as usual $W$ is a 2-dimensional vector subspace. Then by the coset property we know that $\mathbf{z}+W = \mathbf{y}+W = \mathbf{x}+W$, and furthermore we also have $\mathbf{x}+V,\ \mathbf{y}+W \;\subset\; P \;=\; \mathbf{y}+W \;=\; \mathbf{x}+W$. The latter implies that $U$ and $V$ are contained in $W$. Now $\dim W = 2$, and $U \neq V$ implies $\mathbf{u}$ and $\mathbf{v}$ are linearly independent vectors, so $\{\mathbf{u}, \mathbf{v}\}$ forms a basis for $W$.

Returning to the problem of finding a common point for the two lines, we need to find scalars $t$ and $t'$ such that

$$\mathbf{x} \;+\; t \cdot \mathbf{u} \;=\; \mathbf{y} \;+\; t' \cdot \mathbf{v}\ .$$

Now $\mathbf{x}\ , \mathbf{y} \;\in\; P = \mathbf{z} + W$ implies that we can write

$$\mathbf{x} \;=\; \mathbf{z} \;+\; p\,\mathbf{u} \;+\; q\,\mathbf{v}\ , \qquad \mathbf{y} \;=\; \mathbf{z} \;+\; r\,\mathbf{u} \;+\; s\,\mathbf{v}$$

for suitably chosen scalars $p, q, r, s$. We may then reformulate the problem into finding scalars $t$ and $t'$ such that

$$\mathbf{x} \;+\; t \cdot \mathbf{u} \;=\; \mathbf{z} \;+\; (p+t)\,\mathbf{u} \;+\; q\,\mathbf{v} \;=\; \mathbf{z} \;+\; r\,\mathbf{u} \;+\; (s+t')\,\mathbf{v} \;=\; \mathbf{y} \;+\; t' \cdot \mathbf{v}$$

and if we cancel the $\mathbf{z}$ and rearrange terms in the second and third expression we obtain the following equivalent equation:

$$(p-r)\,\mathbf{u} \;+\; (q-s)\,\mathbf{v} \;=\; (-t)\,\mathbf{u} \;+\; t'\,\mathbf{v}$$

Since each of the algebraic steps in this derivation are reversible, it follows that suitable values of $t$ and $t'$ are given by $r - p$ and $q - s$ respectively. This algebraic fact corresponds to the geometric fact that the two lines under consideration have a point in common.■