UNIQUENESS FOR PLANE SEPARATION

Given a plane P and a line $L \subset P$, the **plane separation postulate** states that $P - L$ is a union of two disjoint convex subsets H_1 and H_2 with the following additional property:

If $x_1 \in H_1$ and $x_2 \in H_2$, then there is some point $y \in L$ such that $x_1 * y * x_2$.

We shall prove the following uniqueness result:

Theorem. *Suppose that we have two decompositions $\{H_1, H_2\}$ and $\{H_1', H_2'\}$ of $P - L$ which satisfy the conditions of the plane separation postulate. Then $\{H_1, H_2\} = \{H_1', H_2'\}$.*

There is a similar result for uniqueness in the space separation postulate; the formulation of proof of the latter theorem are left to the reader.

Proof. It suffices to show that *either* $H_1 \subset H_1'$ and $H_2 \subset H_2'$ *or else* $H_1 \subset H_2'$ and $H_2 \subset H_1'$. In fact, it suffices to consider the first case, for the second will then follow by switching the roles of H_1' and H_2'. This is true because we know that $P - L$ is a union of the disjoint subsets H_1 and H_2, and it is also the union of the disjoint subsets H_1' and H_2', for we can switch the roles of the primed and unprimed variables to conclude $H_1 \supset H_2'$ and $H_2 \supset H_1'$.

Again interchanging roles of the variables, we need only show that $H_1 \subset H_1'$ or $H_1 \subset H_2'$. Let $p \in H_1$; then either $p \in H_1'$ or $p \in H_2'$. Once again reversing the roles of variables if necessary, we reduce to considering the case where the first alternative holds.

Since no points of L are in any of the sets $\{H_1, H_2, H_1', H_2'\}$, we must have

$$H_1 \quad = \quad (H_1 \cap H_1') \ \cup \ (H_1 \cap H_2')$$

so it suffices to show that the second summand on the right is empty. Suppose it is not, and let q be a point in this intersection. If we apply the plane separation postulate to $\{H_1', H_2'\}$ we then find that there is a point $z \in L$ such that $p * z * q$. Since $p, q \in H_1$ and the latter is convex, it also follows that $z \in H_1$; this is a contradiction because the sets L and H_1 are disjoint by hypothesis. The source of the contradiction was our supposition that $H_1 \cap H_2'$ was nonempty, so the latter is false and the intersection must indeed be empty.∎