CHAPTER II

AFFINE GEOMETRY

In the previous chapter we indicated how several basic ideas from geometry have natural interpretations in terms of vector spaces and linear algebra. This chapter continues the process of formulating basic geometric concepts in such terms. It begins with standard material, moves on to consider topics not covered in most courses on classical deductive geometry or analytic geometry, and it concludes by giving an abstract formulation of the concept of geometrical incidence and closely related issues.

1. Synthetic affine geometry

In this section we shall consider some properties of Euclidean spaces which only depend upon the axioms of incidence and parallelism

Definition. A *three-dimensional incidence space* is a triple $(S, \mathcal{L}, \mathcal{P})$ consisting of a nonempty set S (whose elements are called *points*) and two nonempty disjoint families of proper subsets of S denoted by $\mathcal{L}$ (*lines*) and $\mathcal{P}$ (*planes*) respectively, which satisfy the following conditions:

(I-1) Every line (element of $\mathcal{L}$) contains at least two points, and every plane (element of $\mathcal{P}$) contains at least three points.

(I-2) If $\mathbf{x}$ and $\mathbf{y}$ are distinct points of S, then there is a unique line L such that $\mathbf{x},\ \mathbf{y} \in L$.

Notation. The line given by **(I-2)** is called the *line joining* $\mathbf{x}$ *and* $\mathbf{y}$ and denoted by $\mathbf{xy}$.

(I-3) If $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are distinct points of S and $\mathbf{z} \notin \mathbf{xy}$, then there is a unique plane P such that $\mathbf{x},\ \mathbf{y},\ \mathbf{z} \in P$.

(I-4) If a plane P contains the distinct points $\mathbf{x}$ and $\mathbf{y}$, then it also contains the line $\mathbf{xy}$.

(I-5) If P and Q are planes with a nonempty intersection, then $P \cap Q$ contains at least two points.

Of course, the standard example in $\mathbb{R}^3$ with lines and planes defined by the formulas in Chapter I (we shall verify a more general statement later in this chapter). A list of other simple examples appears in Prenowitz and Jordan, *Basic Concepts of Geometry*, pp. 141–146.

A few theorems in Euclidean geometry are true for every three-dimensional incidence space. The proofs of these results provide an easy introduction to the synthetic techniques of these notes. In the first six results, the triple $(S, \mathcal{L}, \mathcal{P})$ denotes a fixed three-dimensional incidence space.

Definition. A set B of points in S is *collinear* if there is some line L in S such that $B \subset L$, and it is *noncollinear* otherwise. A set A of points in S is *coplanar* if there is some plane P in S such that $A \subset P$, and it is *noncoplanar* otherwise. — Frequently we say that the points $\mathbf{x},\ \mathbf{y},\ \cdots$ (*etc.*) are collinear or coplanar if the set with these elements is collinear or coplanar respectively.

THEOREM II.1. *Let* $\mathbf{x}$*,* $\mathbf{y}$ *and* $\mathbf{z}$ *be distinct points of* S *such that* $\mathbf{z} \notin \mathbf{xy}$*. Then* $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ *is a noncollinear set.*

Proof. Suppose that L is a line containing the given three points. Since $\mathbf{x}$ and $\mathbf{y}$ are distinct, by **(I-2)** we know that $L = \mathbf{xy}$. By our assumption on L it follows that $\mathbf{z} \in L$; however, this contradicts the hypothesis $\mathbf{z} \notin \mathbf{xy}$. Therefore there is no line containing $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.■

THEOREM II.2. *There is a subset of four noncoplanar points in* S*.*

Proof. Let P be a plane in S. We claim that P contains three noncollinear points. By **(I-1)** we know that P contains three distinct points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}_0$. If these three points are noncollinear, let $\mathbf{c} = \mathbf{c}_0$. If they are collinear, then the line L containing them is a subset of P by **(I-4)**, and since $\mathcal{L}$ and $\mathcal{P}$ are disjoint it follows that L must be a proper subset of P; therefore there is some point $\mathbf{c} \in P$ such that $\mathbf{c} \notin L$, and by the preceding result the set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is noncollinear. Thus in any case we know that P contains three noncollinear points.

Since P is a proper subset of S, there is a point $\mathbf{d} \in S$ such that $\mathbf{d} \notin P$. We claim that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\ \mathbf{d}\}$ is noncoplanar. For if Q were a plane containing all four points, then $\mathbf{a}, \mathbf{b}, \mathbf{c} \in P$ would imply $P = Q$, which contradicts our basic stipulation that $\mathbf{d} \notin P$.■

THEOREM II.3. *The intersection of two distinct lines in* S *is either a point or the empty set.*

Proof. Suppose that $\mathbf{x} \neq \mathbf{y}$ but both belong to $L \cap M$ for some lines L and M. By property **(I-2)** we must have $L = M$. Thus the intersection of distinct lines must consist of at most one point.■

THEOREM II.4. *The intersection of two distinct planes in* S *is either a line or the empty set.*

Proof. Suppose that P and Q are distinct planes in S with a nonempty intersection, and let $\mathbf{x} \in P \cap Q$. By **(I-5)** there is a second point $\mathbf{y} \in P \cap Q$. If L is the line $\mathbf{xy}$, then $L \subset P$ and $L \subset Q$ by two applications of **(I-4)**; hence we have $L \subset P \cap Q$. If there is a point $\mathbf{z} \in P \cap Q$ with $\mathbf{z} \notin L$, then the points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are noncollinear but contained in both of the planes P and Q. By **(I-3)** we must have $P = Q$. On the other hand, by assumption we know $P \neq Q$, so we have reached a contradiction. The source of this contradiction is our hypothesis that $P \cap Q$ strictly contains L, and therefore it follows that $P \cap Q = L$.■

THEOREM II.5. *Let* L *and* M *be distinct lines, and assume that* $L \cap M \neq \varnothing$*. Then there is a unique plane* P *such that* $L \subset P$ *and* $M \subset P$*.*

In less formal terms, given two intersecting lines there is a unique plane containing them.

Proof. Let $\mathbf{x} \in L \cap M$ be the unique common point (it is unique by Theorem 3). By **(I-2)** there exist points $\mathbf{y} \in L$ and $\mathbf{z} \in M$, each of which is distinct from $\mathbf{x}$. The points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are noncollinear because $L = \mathbf{xy}$ and $\mathbf{z} \in M - \{\mathbf{x}\} \ = \ M - L$. By **(I-3)** there is a unique plane P such that $\mathbf{x}, \mathbf{y}, \mathbf{z} \in P$, and by **(I-4)** we know that $L \subset P$ and $M \subset P$. This proves the existence of a plane containing both L and M. To see this plane is unique, observe that every plane Q containing both lines must contain $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$. By **(I-3)** there is a unique such plane, and therefore we must have $Q = P$.■

THEOREM II.6. *Given a line L and a point $\mathbf{z}$ not on L, there is a unique plane P such that $L \subset P$ and $\mathbf{z} \in P$.*

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be distinct points of L, so that $L = \mathbf{xy}$. We then know that the set $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ is noncollinear, and hence there is a unique plane P containing them. By **(I-4)** we know that $L \subset P$ and $\mathbf{z} \in P$. Conversely, if Q is an arbitrary plane containing L and $\mathbf{z}$, then Q contains the three noncollinear points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, and hence by **(I-3)** we know that $Q = P$.■

Notation. We shall denote the unique plane in the preceding result by $L\mathbf{z}$.

Of course, all the theorems above are quite simple; their conclusions are probably very clear intuitively, and their proofs are fairly straightforward arguments. One must add Hilbert's Axioms of Order or the Euclidean Parallelism Axiom to obtain something more substantial. Since our aim is to introduce the parallel postulate at an early point, we might as well do so now (a thorough treatment of geometric theorems derivable from the Axioms of Incidence and Order appears in Chapter 12 of Coxeter, *Introduction to Geometry*; we shall discuss the Axioms of Order in Section VI.6 of these notes).

Definition. Two lines in a three-dimensional incidence space S are *parallel* if they are disjoint and coplanar (note particularly the second condition). If L and L' are parallel, we shall write $L||L'$ and denote their common plane by LL'. — Note that if $L||M$ then $M||L$ because the conditions in the definition of parallelism are symmetric in the two lines.

Affine three-dimensional incidence spaces

Definition. A three-dimensional incidence space $(S, \mathcal{L}, \mathcal{P})$ is an *affine three-space* if the following holds:

(EPP) For each line L in S and each point $\mathbf{x} \notin L$ there is a unique line $L' \subset L\mathbf{x}$ such that $\mathbf{x} \in L$ and $L \cap L' = \varnothing$ (in other words, there is a unique line L' which contains $\mathbf{x}$ and is parallel to L).

This property is often called *the Euclidean Parallelism Property*, the *Euclidean Parallel Postulate* or *Playfair's Postulate*.[1] Actually, Euclid's *Elements* employed a logically equivalent statement which also requires the concepts of betweenness and congruence, and the advantages of using the purely incidence-theoretical statement **(EPP)** were noted explicitly by Proclus Diadochus (412–485).[2]

A discussion of the origin of the term "affine" appears in Section II.5 of the following online site:

`http://math.ucr.edu/~res/math133/geometrynotes2b.pdf`

Many nontrivial results in Euclidean geometry can be proved for arbitrary affine three-spaces. We shall limit ourselves to two examples here and leave others as exercises. In Theorems 7 and 8 below, the triple $(S, \mathcal{L}, \mathcal{P})$ will denote an arbitrary affine three-dimensional incidence space.

THEOREM II.7. *Two lines which are parallel to a third line are parallel.*

[1] JOHN PLAYFAIR (1748–1819) was a Scottish scientist who is also known for an influential book on the philosophy of science; in his geometrical writings, he acknowledged that others had previously considered **EPP** as an axiom for geometry.

[2] The writings of Proclus provide valuable information on important, but apparently lost, works of earlier Greek mathematicians.

Proof. There are two cases, depending on whether or not all three lines lie in a single plane; to see that the three lines need not be coplanar in ordinary 3-dimensional coordinate geometry, consider the three lines in $\mathbb{R}^3$ given by the z-axis and the lines joining $(1,0,0)$ and $(0,1,0)$ to $(1,0,1)$ and $(0,1,1)$ respectively.

THE COPLANAR CASE. Suppose that we have three distinct lines L, M, N in a plane P such that $L||N$ and $M||N$; we want to show that $L||N$.

If L is not parallel to N, then there is some point $\mathbf{x} \in L \cap N$, and it follows that L and N are distinct lines through $\mathbf{x}$, each of which is parallel to M. However, this contradicts the Euclidean Parallel Postulate. Therefore the lines L and N cannot have any points in common.

THE NONCOPLANAR CASE. Let α be the plane containing L and M, and let β be the plane containing M and N. By the basic assumption in this case we have $\alpha \neq \beta$. We need to show that $L \cap N = \varnothing$ but L and N are coplanar.

The lines L and N are disjoint. Assume that the L and N have a common point that we shall call $\mathbf{x}$. Let γ be the plane determined by $\mathbf{x}$ and N (since $L||M$ and $\mathbf{x} \in L$, clearly $\mathbf{x} \notin M$). Since $\mathbf{x} \in L \subset \alpha$ and $M \subset \alpha$, Theorem 6 implies that $\alpha = \gamma$. A similar argument shows that $\beta = \gamma$ and hence $\alpha = \beta$; the latter contradicts our basic stipulation that $\alpha \neq \beta$, and therefore it follows that L and N cannot have any points in common.

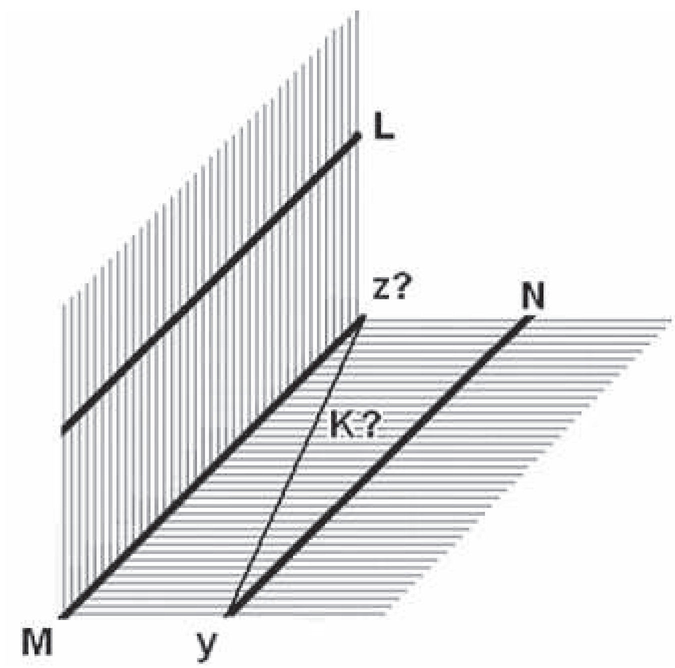

Figure II.1

The lines L and N are coplanar. Let $\mathbf{y} \in N$, and consider the plane $L\mathbf{y}$. Now L cannot be contained in β because $\beta \neq \alpha = LM$ and $M \subset \beta$. By construction the planes $L\mathbf{y}$ and β have the point $\mathbf{y}$ in common, and therefore we know that $L\mathbf{y}$ meets β in some line K. Since L and K are coplanar, it will suffice to show that $N = K$. Since N and K both contain $\mathbf{y}$ and all three lines M, N and K are contained in β, it will suffice to show that $K||M$.

Suppose the lines are not parallel, and let $\mathbf{z} \in K \cap M$. Since $L||M$ it follows that $\mathbf{z} \notin L$. Furthermore, $L \cup K \subset L\mathbf{y}$ implies that $\mathbf{z} \in L\mathbf{y}$, and hence $\mathbf{y} = L\mathbf{z}$. Since $\mathbf{z} \in M$ and L and M are coplanar, it follows that $M \subset L\mathbf{z}$. Thus M is contained in $L\mathbf{y} \cap \beta$, and since the latter is the line K, this shows that $M = K$. On the other hand, by construction we know that $M \cap N = \varnothing$ and $K \cap N \neq \varnothing$, so that M and K are obviously distinct. This contradiction implies that $K||M$ must hold.■

The next result is an analog of the Parallel Postulate for parallel planes.

THEOREM II.8. *If P is a plane and $\mathbf{x} \notin P$, then there is a unique plane Q such that $\mathbf{x} \in Q$ and $P \cap Q = \varnothing$.*

Proof. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in P$ be the noncollinear points, and consider the lines A', B' through $\mathbf{x}$ which are parallel to $A = \mathbf{bc}$ and $B = \mathbf{ac}$. Let Q be the plane determined by A' and B', so that $\mathbf{x} \in Q$ by hypothesis. We claim that $P \cap Q = \varnothing$.

Assume the contrary; since $\mathbf{x} \in QA$ and $\mathbf{x} \notin P$, the intersection $P \cap Q$ is a line we shall call L.

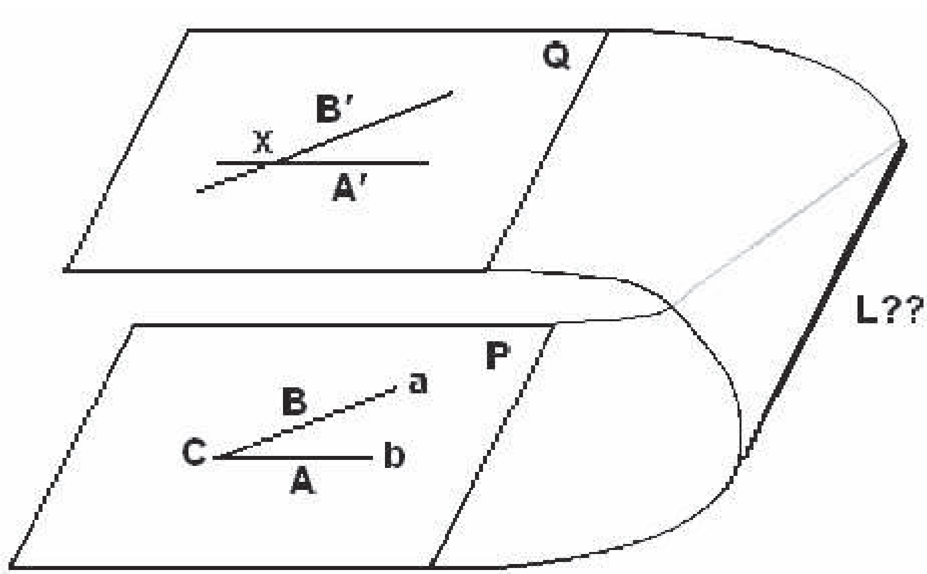

Figure II.2

STEP 1. We shall show that $L \neq A,\ B$. The proof that $L \neq A$ and $L \neq B$ are similar, so we shall only show $L \neq A$ and leave the other part as an exercise. — If $L = A$, then $L \subset Q$. Since A' is the unique line in Q which is parallel to A, there must be a point $\mathbf{u} \in B' \cap A$. Consider the plane $B'\mathbf{c}$. Since $\mathbf{c} \in A$, it follows that $A \subset B'\mathbf{c}$. Hence $B'\mathbf{c}$ is a plane containing A and B. The only plane which satisfies these conditions is P, and hence $B' \subset P$. But $\mathbf{x} \in B'$ and $\mathbf{x} \notin P$, so we have a contradiction. Therefore we must have $L \neq A$.

STEP 2. We claim that either $A' \cap L$ and $A \cap L$ are both nonempty or else $B' \cap L$ and $B \cap L$ are both nonempty. — We shall only show that if either $A' \cap L$ is $A \cap L$ is empty then both $B' \cap L$ and $B \cap L$ are both nonempty, since the other case follows by reversing the roles of A and B. Since L and A both lie in the plane P, the condition $A \cap L = \varnothing$ implies $A||L$. Since $A||A'$, by Theorem 7 and Theorem 1 we know that $A'||L$. Since $B \neq A$ is a line through the point $\mathbf{c} \in A$, either $B = L$ or $B \cap L \neq \varnothing$ holds by the Parallel Postulate (in fact, $B \neq L$ by Step

1). Likewise, B and B' are lines through $\mathbf{x}$ in the plane Q and $L \subset Q$, so that the $A'||L$ and the Parallel Postulate imply $B' \cap L \neq \emptyset$.

STEP 3. There are two cases, depending upon whether $A' \cap L$ and $A \cap L$ are both nonempty or $B' \cap L$ and $B \cap L$ are both nonempty. Only the latter will be considered, since the former follows by a similar argument. Let $\mathbf{y} \in B \cap L$ and $\mathbf{z} \in B' \cap L$; since $B \cap B' = \emptyset$, if follows that $\mathbf{y} \neq \mathbf{z}$ and hence $L = \mathbf{yz}$. Let β be the plane BB'. Then $L \subset \beta$ since $\mathbf{z},\ \mathbf{y} \in \beta$. Since $L \neq B$, the plane β is the one determined by L and B. But $L,\ B \subset P$ by assumption, and hence $\beta = P$. In other words, B' is contained in P. But $\mathbf{x} \in B'$ and $\mathbf{x} \notin P$, a contradiction which shows that the line L cannot exist.■

Following standard terminology, we shall say that the plane Q is *parallel* to P or that it is the plane parallel to P which passes through $\mathbf{x}$.

Corresponding definitions for incidence planes and affine planes exist, and analogs of Theorems 1, 2, 3 and 7 hold for these objects. However, incidence planes have far fewer interesting properties than their three-dimensional counterparts, and affine planes are best studied using the methods of projective geometry that are developed in later sections of these notes.

EXERCISES

Definition. A line and a plane in a three-dimensional incidence space are *parallel* if they are disjoint.

Exercises **1–4** *are to be proved for arbitrary* 3*-dimensional incidence spaces.*

1. Suppose that each of two intersecting lines is parallel to a third line. Prove that the three lines are coplanar.

2. Suppose that the lines L and L' are coplanar, and there is a line M not in this plane such that $L||M$ and $L'||M$. Prove that $L||L'$.

3. Let P and Q be planes, and assume that each line in P is parallel to a line in Q. Prove that P is parallel to Q.

4. Suppose that the line L is contained in the plane P, and suppose that $L||L'$. Prove that either L' is parallel to P or else $L \subset P$.

In exercises **5–6**, *assume the incidence space is affine.*

5. Let P and Q be parallel planes, and let L be any line which contains a point of Q and is parallel to a line in P. Prove that L is contained in Q. [*Hint:* Let M be the line in P, and let $\mathbf{x} \in L \cap Q$. Prove that $L = M\mathbf{x} \cap Q$.]

6. Two lines are said to be **skew lines** if they are not coplanar. Suppose that L and M are skew lines. Prove that there is a unique plane P such that $L \subset P$ and P is parallel to M. [*Hint:* Let $\mathbf{x} \in L$, let M' be a line parallel to M which contains $\mathbf{x}$, and consider the plane LM'.]