Geometric transformations and the Two Circle Theorem

One major advantage of geometric transformations is that they can often be used to simplify gemetric proofs, reducing them to the consideration of relatively simple examples; the file `elltangents.pdf` proves on specific result from this viewpoint. In this document we shall apply the same idea to give a slightly different proof of the Two Circle Theorem (III.6.3 in the course notes). Since isometries preserve distances, lines and the half-planes determined by lines, if the theorem is valid for a specific pair of circles Γ_1 and Γ_2 then for every isometry T the theorem is also valid for the circles $T[\Gamma_1]$ and $T[\Gamma_2]$. Therefore we can organize the proof of the Two Circle Theorem around two steps:

(1) Given two circles Γ_1 and Γ_2 with centers z_1 and z_2 respectively, there is an isometry of $\mathbb{R}^2$ taking z_1 to $(0,0)$ and z_2 to $(d,0)$ for some $d>0$ (note that $z_1 \neq z_2$ because two circles with the same center are either disjoint or identical).

(2) Show the validity of the result when Γ_1 is the circle with center $(0,0)$ and defining equation $x^2+y^2=a^2$ and Γ_2 is the circle with center $(d,0)$ and defining equation $(x-d)+y^2=b^2$ for suitable $a,b,d>0$.

We can verify the first part as follows. There is a translation sending z_1 to $(0,0)$, and there is a rotation T_2 sending $T_1(z_2)$ to a point on the positive x-axis. If $T=T_1 \circ T_2$, then T sends z_1 to $(0,0)$ and z_2 to $(d,0)$ for some $d>0$.

To verify the second step when $z_1=(0,0)$ and $z_2=(d,0)$, it is only necessary to rewrite everything in terms of coordinates. The equations defining Γ_1 and Γ_2 are $x^2+y^2=a^2$ and $(x-d)^2+y^2=b^2$ respectively, and it follows that if $(x,y)\in\Gamma_2$ then the second equation can be rewritten in the form

$$x^2 + y^2 = b^2 + 2xd - d^2 = \rho(x) .$$

Now the hypotheses of the Two-Circle Theorem imply that for some choice (x_1,y_1) of (x,y) on Γ_2 we have $\rho(x_1)<a^2$ and for some choice (x_2,y_2) of (x,y) on Γ_2 we have $\rho(x_2)>a^2$. Since a point $(u,v)\in\Gamma$ satisfies

$$d - b \leq u \leq d + b$$

the function ρ takes its minimum at $d-b$ and its maximum at $d+b$, so that

$$\rho(d-b) \leq \rho(x_1) < a^2 < \rho(x_2) < \rho(d+b)$$

and if we evaluate the outside terms explicitly we obtain the following inequalities:

$$b^2 - 2db + d^2 = b^2 + 2d(d-b) - d^2 < a^2 <$$

$$b^2 + 2d(d+b) - d^2 = b^2 + 2bd - b^2$$

Taking square roots, we see that the displayed inequalities are equivalent to $|b-d| < a < d+b$.

If a point (x,y) lies on $\Gamma_1\cap\Gamma_2$, then the preceding discussion implies that $a^2=\rho(x)=b^2-2dx-d^2$; solving for x, we find that

$$x = \frac{a^2+d^2-b^2}{2d}$$

and $y = \pm\sqrt{a^2 - x^2}$. In order to verify that we actually obtain two points, we need to check that $|x| < a$. But the latter is equivalent to $-2da \ < \ a^2 + d^2 - q^2 \ < \ 2da$, which in turn is equivalent to each of the next four lines:

$$-a^2 \ - \ 2da \ - \ d^2 \quad < \quad -b^2 \quad < \quad -a^2 \ + \ 2da \ - \ d^2$$

$$-(a+d)^2 \quad < \quad -b^2 \quad < \quad -(a-d)^2$$

$$(a-d)^2 \quad < \quad b^2 \quad < \quad (a+d)^2$$

$$|a-d| \quad < \quad b \quad < \quad a+d$$

Therefore it suffices to prove that the last chain of equalities is valid.

By earlier steps in the argument we know that $|b-d| \ < \ a \ < \ b+d$. Now $a < b+d$ implies $a-d < b$, while $b-d < a$ implies $b < a+d$ and $d-b < a$ implies $d-a < p$. These inequalities combine to prove the inequalities in the last of the displayed lines, and therefore the desired inequality $|x| < a$ follows, which means that $\Gamma_1 \cap \Gamma_2$ consists of two points, with one on each side of the x-axis.■

Finally, we note that the algebraic part of the preceding proof is the same as the algebraic part of the proof for Theorem III.6.3 in the course notes except that (p,q) in the notes corresponds to (a,b) in this document.