Readings for Mathematics 133, Fall 2013, from Stillwell, *The Four Pillars of Geometry*

General remarks

This book presents geometry as a subject which can be approached logically from four separate directions

1. The ***classical Greek*** (synthetic) ***approach.***
2. The (analytic) approach using ***vector geometry and linear algebra.***
3. The ***projective — geometric approach***, which provides a unified viewpoint for studying many topics and problems.
4. The use of ***geometric transformations.***

Forms of all four approaches are present in the files of course notes. However, the third approach — which is discussed at length in Unit **IV** of the course notes — will not be covered in the course itself. Furthermore, the fourth approach receives far less attention than the synthetic and analytic approaches, and this is mainly restricted to a discussion of congruence and similarity for geometrical figures.

The title of the book is derived from a Chinese (also Japanese and Korean) concept

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involving data that are supposed to shape an individual's destiny; the ***Wikipedia*** article

http://en.wikipedia.org/wiki/Four_Pillars_of_Destiny

contains (unchecked) further information on this topic. Some comments in the Preface to Stillwell's book reflect some similarities between his book and the viewpoint of this course:

> Geometry, of all subjects, should be about *taking different viewpoints*, and geometry is unique among the mathematical disciplines in its ability to look different from different angles [not everyone would agree with the assertion of uniqueness]. Some prefer to approach it visually, others algebraically, but … they are all looking at the same thing. …
>
> The many faces of geometry are [potentially] … a great help to the learner and teacher. We all know that some students prefer to visualize, whereas others prefer to reason or to calculate. Geometry has something for everybody, and all students will find themselves building on their strengths at some times, and working to overcome weaknesses at other times. We also know that Euclid has some beautiful proofs, whereas other theorems are more beautifully proved by algebra. In the multifaceted approach, every theorem can be given an elegant [*or at least the most transparent*] proof, and theorems with radically different proofs can be viewed from different sides.

Readings for Unit I from Stillwell
(Topics from linear algebra)

I.0 : Background

Suggested readings.

Stillwell : Sections **3.1, 4.1, 4.2, 4.8**

I.1 : Dot products

Suggested readings.

Stillwell : Sections **3.1, 3.5, 4.4, 4.5, 4.6**

I.2 : Cross products

Suggested readings.

[None]

I. 3 : Linear varieties

Suggested readings.

Stillwell : Section **3.2**

I.4 : Barycentric coordinates

Suggested readings.

Stillwell : Section **4.3**

Readings for Unit II from Stillwell
(Vector algebra and Euclidean geometry)

II.1 : Approaches to Euclidean geometry

Suggested readings.

Stillwell : Sections **2.9, 3.2**

II.2 : Synthetic axioms of order and separation

Suggested readings.

Stillwell : Sections **2.9, 3.2, 3.3**

II.3 : Measurement axioms

Suggested readings.

Stillwell : Sections **2.2, 2.9, 3.5**

II.4 : Congruence, superposition and isometries

Suggested readings.

Stillwell : Sections **2.2, 2.9, 3.6, 3.7, 4.7, 7.1, 7.2, 7.9**

Comments.

The material in Section **3.7** is at a higher level than that of the course and will not be covered; a reference is included because the topic may be of interest to some readers.

II.5 : Euclidean parallelism

Suggested readings.

Stillwell : Sections **2.1, 2.9, 7.9**

Readings for Unit III from Stillwell
(Basic Euclidean concepts and theorems)

III.1 : Perpendicular lines and planes

Supplementary background readings.

[None]

Comments.

Perpendicularity is first defined in Section **1.3** of Stillwell using classical geometrical constructions.

III.2 : Basic theorems on triangles

Supplementary background readings.

Stillwell : Sections **2.1, 2.2, 2.5, 2.8, 3.3**

Comments.

The proof of the Pythagorean Theorem in Section **2.5** of Stillwell involves areas (which are not covered until Section **III.7** in this course), and this proof is entirely different from the one in the course notes. There are many proofs that are much shorter and easier to follow, even if one uses area theory; for example, the drawing which appears in the file **http://math.ucr.edu/~res/math153/pythagorean-thm.pdf** is clearly much simpler than the one considered in Section **2.5** of Stillwell. The latter proof appears in the ***Elements***; although the value of the proof is mainly historic, the presentation in Stillwell provides a particularly clear overview of the argument. In a purely synthetic but logically rigorous approach to elementary geometry, the simplest way of proving the Pythagorean Theorem is probably by means of similar triangles along the lines of the argument in Section **2.8** of Stillwell (in the course notes, similar triangles are discussed in Section **III.5**).

III.3 : Convex polygons

Supplementary background readings.

Stillwell : Sections **2.2, 4.3**

Comments.

The discussion of quadrilaterals in Stillwell is almost entirely limited to a few results about parallelograms.

III.4 : Concurrence theorems

Supplementary background readings.

Stillwell : Sections **2.1, 2.9, 3.3, 4.3**

Comments.

A very instructive proof for a $3-$ dimensional analog of the Triangle Centroid Theorem (Theorem **III.4.1**) is outlined in the Exercises for Section **4.3** of Stillwell.

III.5 : Similarity

Supplementary background readings.

Stillwell : Sections **1.5, 2.6, 2.8**

Comments.

The proof of the basic similarity theorem (called Thales' Theorem in Stillwell) involves areas, and as such it is entirely different from the proof of the corresponding result in the course notes (see Theorem **III.5.10**). A proof of the Pythagorean Theorem using similar triangles is described in Section **2.8** of Stillwell.

III.6 : Circles and constructions

Supplementary background readings.

Stillwell : Sections **1.1, 1.2, 1.3, 1.4, 1.6, 2.7, 3.4, 3.8**

Comments.

The treatment in Stillwell places a great deal of emphasis on classical geometrical constructions, while the latter are only mentioned fairly briefly in the course notes. Also, the main result in Section **2.7** of Stillwell, which deals with the measures of an angle inscribed in a circle and its intercepted arc, is mentioned (with an illustration) on page **137** of the course notes (see the file **http://math.ucr.edu/~res/math133/semicircle-thm.pdf** for a special case). Finally, the main results of Section **3.4** of Stillwell are covered in Section **III.6** of the course notes.

Section **3.8** of Stillwell contains a few remarks about ***algebraic geometry***, a subject which originated in the study of curves in the coordinate plane which are defined by solutions to polynomial equations in the coordinates. Circles are obvious examples of curves defined by second degree polynomials in two variables, and the latter class also includes the standard conic section curves. The following two comments on this topic give some further information:

1. Stillwell gives two references for further information about algebraic geometry (namely, Brieskorn — Knörrer, which Stillwell translated into English very successfully, and McKean — Moll). Other books on this topic at the advanced undergraduate level include **(1)** G. Fischer, _Plane Algebraic Curves_, **(2)** W. Jenner, _Rudiments of Algebraic Geometry_, **(3)** M. Reid, _Undergraduate Algebraic Geometry_, and **(4)** A. Seidenberg, _Elements of the Theory of Algebraic Curves_ (this list is not meant to be exhaustive). For most of these books, the material becomes considerably more challenging after the first one or two chapters. In particular, this statement applies to the discussion of third degree curves in each book and most notably to McKean — Moll, which is really a study of such curves rather than an introduction to algebraic geometry in general. The subject of third degree curves has been studied extensively over the past three centuries, it has seen many very remarkable discoveries, and it has an extremely wide range of applications to subjects from modern theoretical physics to coding theory.

2. The assertion that linear algebra is contained in algebraic geometry is oversimplified and ultimately misleading. As with many topics in mathematics, both of these subjects have grown, and to a great extent they have gone their separate ways; unfortunately, a detailed explanation of the latter quickly gets far beyond the level of this course.

III.7 : Areas and volumes

Supplementary background readings.

Stillwell : Sections **2.1, 2.3, 2.4, 2.5. 2.6**

Comments.

Stillwell's treatment of area is basically informal and includes historical background from Greek geometry. As noted previously, at a few points Stillwell uses considerations involving area to derive some fundamental theorems in Euclidean geometry.

Readings for Unit IV from Stillwell (Projective geometry)

Since this material is not covered in the course, there are only a few comments to compare and contrast the material in Stillwell and the course notes.

IV.1 : Perspective images

Supplementary background readings.

Stillwell : Sections **5.1, 5.2, 5.5, 7.9**

Comment.

Among other things, Section **7.9** of Stillwell discusses an alternative to the standard method for representing parallel lines which is central to the theory of perspective drawing; the approach is taken from 18th century Japanese art. However, no evidence is presented to support an assertion on page 170 of Stillwell that affine geometry is derived from Chinese and Japanese drawing techniques.

IV.2 : Adjoining ideal points

Suggested readings.

Stillwell : Sections **5.3, 5.4**

IV.3 : Homogeneous coordinates

Suggested readings.

Stillwell : Sections **5.3, 5.4, 5.9, 6.4 – 6.7**

IV.4 : Projective duality

Suggested readings.

Stillwell : Section **5.3**

Comment.

The concept of duality in projective geometry does not appear explicitly in Stillwell.

IV.5 : Theorems of Desargues and Pappus

Suggested readings.

Stillwell : Sections **5.4, 5.9, 6.1 – 6.3, 6.8**

IV.6 : Cross ratios and projective collineations

Suggested readings.

Stillwell : Sections **5.4 – 5. 9, 7.3**

Comment.

The cross ratio plays an important role in each of the models for non – Euclidean geometry which appear subsequently in the Unit **V.**

Readings for Unit V from Stillwell
(Introduction to non – Euclidean geometry)

V.1 : Facts from spherical geometry

Supplementary background readings.

Stillwell : Sections **7.4, 7.5, 7.6, 7.8, 8 - Preview, 8.5**

Comments. For the most part, the discussion of spherical geometry in Stillwell is based upon a study if the isometries of the sphere, which are all given by 3×3 orthogonal matrices. Here are further comments on two points in Stillwell:

1. In the Preview to Chapter 8 there is a statement that spherical geometry "was never seen as a challenge to Euclid, probably because the geometry is simply a part of three – dimensional Euclidean geometry, where great circles coexist with genuine straight lines." This may be partly true, but other plausible explanations have been given and are more widely accepted. One of the most widely accepted views is that spherical geometry also does not satisfy another of Euclid's axioms; namely, lines extend indefinitely in each direction such that the distance from a given point can be arbitrarily large (Euclid's Second Postulate). In the work on non – Euclidean geometry before Riemann's penetrating insights, mathematicians used the Second Postulate to exclude systems which somehow resembled spherical geometry. Similarities between hyperbolic and spherical geometry had been noted, but only in passing and without serious efforts to determine if they were more than coincidental.
2. In Section **7.6** the algebra of quaternions is described as "the most elegant (and practical) way to describe rotations'' in the sphere or in $3-$ space. This certainly was true in the second half of the 19^{th} century and it is still valid in many important respects, but the assertion should probably be less absolute, and a person who is studying such rotations for the first time might find it better to have an introduction which is more elementary and less algebraic in nature. However, at the graduate level it is necessary to understand the relationship between quaternions and $3-$ dimensional rotations.

In a less controversial direction, the exercises for Section **8.5** of Stillwell sketch a proof that the area of a spherical triangle is proportional to the excess of its angle sum over 180 degrees (Theorem $\mathbf{V.1.3}$); Stillwell also notes that this result is (independently) due to the English mathematician T. Harriot (1560 – 1621).

V.2 : Attempts to prove Euclid's Fifth Postulate

V.3 : Neutral geometry

V.4 : Angle defects and related phenomena

V.5 : Further topics in hyperbolic geometry

Suggested readings.

Stillwell : Section **8. 9**

Comments. The approaches to non – Euclidean geometry in Stillwell and the course notes are entirely opposite in nature. In the course notes, the approach roughly follows the historical path leading to the development of hyperbolic geometry by mainly synthetic means. Stillwell's approach starts at the other end, beginning with a standard mathematical model for the subject which was developed near or after the final realization that Euclid's Fifth Postulate could not be derived from the remaining assumptions (in a modern, more rigorous form) and limiting the discussion of the synthetic approach to a very limited number of comments in Section **8.9.** Describing and working with the standard models both require a substantial amount of background material which include topics normally taught in a complex variables course (like Mathematics **165AB**) and other material at an equivalent level. On the other hand, in a completely synthetic approach it is at least somewhat difficult to study some basic phenomena in hyperbolic geometry such as the (critical) angle of parallelism and asymptotic parallel lines.

In Section **8.9** Stillwell suggests one way of viewing the two types of hyperbolic parallel lines in a unified framework which reflects some observations of Saccheri: If two lines are parallel but not critically parallel in hyperbolic geometry, then they have a unique common perpendicular; if two lines are critically parallel, then they approach each other asymptotically at one (common) end and one can view them as having a common perpendicular at infinity.

V.6 : Subsequent developments

Supplementary background readings.

Stillwell : Sections **5.6, 8.1 – 8.6, 8. 9**

Comments.

The material in Section **V.6** ends with comments about standard mathematical models for hyperbolic geometry, and in contrast the treatment of hyperbolic geometry in Chapter

8 of Stillwell begins with the construction of a model. This model — the so – called ***Poincaré half – plane model*** which is introduced near the beginning of Section **V.7** in the notes — is different from either of the two models discussed in this section of the notes, and it has several advantages. For example, several properties of the hyperbolic plane are particularly easy to visualize (in particular, see pp. **176 – 177** of Stillwell). The definition of distances for this model is given somewhat indirectly in Stilwell. If two points lie on the same vertical line, then the definition is given explicitly on page **193** of Stillwell, and the definition is extended to more general cases by means of geometrical transformations. This extension process relies on material developed earlier in Chapter **8.** The file **http://cs.unm.edu/~joel/NonEuclid/model.html** gives an explicit definition of distance for all pairs of points in the half – plane model. As in Stilwell's discussion, the general definition of distance is related to concepts from Unit **IV** in the notes (but none of this is needed in the course). The model for hyperbolic geometry which is described in this section of the notes (the Beltrami model) is discussed on pages **206 – 210** of Stillwell.

V.7 : Non – Euclidean geometry in modern mathematics

Supplementary background readings.

Stillwell : Sections **5.6, 7. 7, 7.8, 8 - Preview, 8.1 – 8.6, 8.8, 8. 9**

Comments.

Since the role of hyperbolic geometry in modern mathematics depends heavily on detailed studies of the standard models, it is not surprising that the treatments in Stillwell and the course are much closer in this section than in Sections **V.2 – V.6.** The discussion below will concentrate on the references in Stillwell for various points covered in Section **V.7** of the notes.

Stillwell's development of the half – plane model for hyperbolic geometry, which is done mainly in Sections **8.1** and **8.2,** gives very effective and relatively simple illustrations of many basic phenomena in hyperbolic geometry; in the discussion of the preceding section of the course notes, we gave references for the definition of distance in that model. The Poincaré disk model for hyperbolic geometry is discussed on pages **209 – 210** of Stillwell.

One major theme in this section of the notes involves the large number of ways in which one can decompose the hyperbolic plane into regular polyhedra. Such decompositions are discussed in Section **8.5** and pages **210 – 212** of Stillwell. By construction these structures are very symmetric (just like their Euclidean counterparts), so the study of regular decompositions has close ties to the study of certain types of geometric isometries. The latter are mentioned on the relevant pages of Stillwell, and some examples are also discussed in Section **8.7** of Stillwell (with some background information in Section **8.1**).