Solutions for `quiz2f20.pdf`

1. This is a straightforward application of the Angle Bisector Theorem: We have a right triangle whose vertices are $X = (p, q)$, $Y = (0, 0)$ and $Z = (25, 0)$ where p and q are specific positive integers whose sum is less than or equal to 18. If $W = (w, 0)$ denotes the point of interest, then the Bisector Theorem then says that

$$\frac{|XY|}{|XZ|} \quad = \quad \frac{w}{25 - w}$$

and by the hypotheses we know that $|XY| = \sqrt{p^2 + q^2}$ and $|YZ| = \sqrt{(25 - p)^2 + q^2}$. Solving for w is a straightforward exercise in algebra.■

2. If θ is the measure of the angle described at the beginning of the problem, then we know that $m = \tan \theta$. Furthermore, the bisection hypothesis implies that $k = \tan(\theta/2)$. Thus the goal of the problem is to find a formula for $\tan(\theta/2)$ in terms of $\tan \theta$. If we follow the hint, we find the following formula for $\tan \theta = m$ in terms of $\tan(\theta/2) = k$ in the suggested *Wikipedia* article:

$$\tan \alpha \quad = \quad \frac{2 \tan(\alpha/2)}{1 - \tan^2(\alpha/2)} \qquad \text{or equivalently} \qquad m \quad = \quad \frac{2k}{1 - k^2}$$

This gives us m in terms of k, but we really want to find k in terms of m. The first step is to clear the second equation of fractions:

$$m \; - \; k^2 m \quad = \quad 2k \qquad \text{or equivalently} \qquad mk^2 \; + \; 2k \; - \; m \quad = \quad 0$$

If we apply the Quadratic Formula to this we obtain

$$k \quad = \quad \frac{-2 \pm \sqrt{4 + 4m^2}}{2m}$$

and since this has two roots we need to determine which one gives the correct answer to the problem. Now $4 + 4m^2 \geq 4$, and therefore the right hand side is positive for $-2 + \sqrt{4 + 4m^2}$ and negative for $-2 - \sqrt{4 + 4m^2}$. Therefore the correct answer to the problem is

$$k \quad = \quad \frac{(-2) \; + \; \sqrt{4 + 4m^2}}{2m} \; .■$$