Exercises for Unit I
(Topics from linear algebra)

I.0 : Background

Note. There is no corresponding section in the course notes, but as noted at the beginning of Unit **I** these are a few exercises which involve the prerequisites from linear algebra; most if not all of this material will be used later in the course.

1. Suppose that **V** is a vector space and that **x** and **y** are nonzero vectors in **V.** Prove that the set $\{\mathbf{x}, \mathbf{y}\}$ is linearly dependent if and only if **x** and **y** are nonzero multiples of each other.

2. Let **V** be a vector space, let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of linearly independent vectors in **V,** and let **W** be the subspace spanned by S. Suppose that $\mathbf{z}$ is a vector in **V** which does not lie in **W.** Prove that the set $S \cup \{\mathbf{z}\}$ is linearly independent.

3. Let **V** be a vector space, let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of linearly independent vectors in **V,** and let $\{c_1, c_2, \ldots, c_k\}$ be a sequence of nonzero scalars. Prove that S is linearly independent if and only if the set $\{c_1\mathbf{v}_1, c_2\mathbf{v}_2, \ldots, c_k\mathbf{v}_k\}$ is linearly independent.

4. Let **V** and **W** be vector spaces, let $\mathbf{T} : \mathbf{V} \to \mathbf{W}$ be a linear transformation which is invertible, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a finite subset of vectors in **V.** Prove that S is linearly independent if and only if the set $\mathbf{T}[S] = \{\mathbf{T}(\mathbf{v}_1), \mathbf{T}(\mathbf{v}_2), \ldots, \mathbf{T}(\mathbf{v}_k)\}$ is.

I.1 : Dot products

1. Compute the dot product $\mathbf{a} \cdot \mathbf{b}$, where $\mathbf{a} = (3, 4)$ and $\mathbf{b} = (2, -3)$.

2. Compute the dot product $\mathbf{a} \cdot \mathbf{b}$, where $\mathbf{a} = (2, -3, 4)$ and $\mathbf{b} = (0, 6, 5)$.

3. Compute the dot product $\mathbf{a} \cdot \mathbf{b}$, where $\mathbf{a} = (2, -1, 1)$ and $\mathbf{b} = (1, 0, -1)$.

4. Determine whether the vectors $\mathbf{a} = (4, 0)$ and $\mathbf{b} = (1, 1)$ are perpendicular, linearly dependent, or neither.

5. Determine whether the vectors $\mathbf{a} = (2, 18)$ and $\mathbf{b} = (9, -1)$ are perpendicular, linearly dependent, or neither.

6. Determine whether the vectors $\mathbf{a} = (2, -3, 1)$ and $\mathbf{b} = (-1, -1, -1)$ are perpendicular, linearly dependent, or neither.

7. Consider a regular tetrahedron **T** (a pyramid with triangular base, where all faces are equilateral triangles) whose vertices are $(0, 0, 0)$, $(k, k, 0)$, $(k, 0, k)$, and $(0, k, k)$ for some positive constant k. Find the degree measure of the angle $\angle xzy$, where z is the centroid of **T** — whose coordinates are $(\frac{1}{2}k, \frac{1}{2}k, \frac{1}{2}k)$ — and x and y are any two vertices (the answer will be the same for all choices).

8. Given the vectors $\mathbf{u} = (2, 3)$ and $\mathbf{v} = (5, 1)$, write $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1$, where $\mathbf{u}_0$ is a scalar multiple of $\mathbf{v}$ and $\mathbf{u}_1$ is perpendicular to $\mathbf{v}$.

9. Given the vectors $\mathbf{u} = (2, 1, 2)$ and $\mathbf{v} = (0, 3, 4)$, write $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1$, where $\mathbf{u}_0$ is a scalar multiple of $\mathbf{v}$ and $\mathbf{u}_1$ is perpendicular to $\mathbf{v}$.

10. Given the vectors $\mathbf{u} = (5, 6, 2)$ and $\mathbf{v} = (-1, 3, 4)$, write $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1$, where $\mathbf{u}_0$ is a scalar multiple of $\mathbf{v}$ and $\mathbf{u}_1$ is perpendicular to $\mathbf{v}$.

11. Given the vectors $\mathbf{u} = (-1, 1, 1)$ and $\mathbf{v} = (2, 1, -3)$, write $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1$, where $\mathbf{u}_0$ is a scalar multiple of $\mathbf{v}$ and $\mathbf{u}_1$ is perpendicular to $\mathbf{v}$.

12. Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in the real inner product space $\mathbb{R}^n$ such that $\mathbf{u}\cdot\mathbf{v} = 2$, $\mathbf{v}\cdot\mathbf{w} = -3$, $\mathbf{u}\cdot\mathbf{w} = 5$, $||\mathbf{u}|| = 1$, $||\mathbf{v}|| = 2$, and $||\mathbf{w}|| = 7$. Evaluate the following expressions:

(a) $(\mathbf{u} + \mathbf{v})\cdot(\mathbf{v} + \mathbf{w})$
(b) $(2\mathbf{v} - \mathbf{w})\cdot(3\mathbf{u} + 2\mathbf{w})$
(c) $(\mathbf{u} - \mathbf{v} - 2\mathbf{w})\cdot(4\mathbf{u} + \mathbf{v})$
(d) $||\mathbf{u} + \mathbf{v}||$
(e) $||2\mathbf{w} - \mathbf{v}||$
(f) $||\mathbf{u} - 2\mathbf{v} + 4\mathbf{w}||$

13. Apply the Gram – Schmidt orthogonalization process to the following vectors in $\mathbb{R}^n$ with the standard scalar product:

(a) $\mathbf{v}_1 = (1, 1, 0)$, $\mathbf{v}_2 = (0, 1, 1)$, $\mathbf{v}_3 = (1, 1, 1)$

(b) $\mathbf{v}_1 = (1, 0, 0, 0)$, $\mathbf{v}_2 = (1, 1, 0, 1)$, $\mathbf{v}_3 = (1, 1, 1, 0)$, $\mathbf{v}_4 = (1, 1, 1, 1)$

(c) $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 1, 0)$, $\mathbf{v}_3 = (-1, -1, 1)$

14. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis of the real inner product space **V**. Show that for every vector **w** in **V** one has the identity

$$||\mathbf{w}||^2 = \langle \mathbf{w}, \mathbf{v}_1\rangle^2 + \langle \mathbf{w}, \mathbf{v}_2\rangle^2 + \ldots + \langle \mathbf{w}, \mathbf{v}_n\rangle^2.$$

15. Let **W** be the subspace of $\mathbb{R}^3$ spanned by $(1, 2, -1)$.

(a) Find an explicit formula for the orthogonal projection onto **W** (with respect to the standard scalar product).

(b) Find the matrix representation of this projection with respect to the standard basis of unit vectors.

16. Suppose that two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in the inner product space $\mathbf{V}$ are orthogonal and satisfy $||\mathbf{x}|| = ||\mathbf{y}||$. Show that $\mathbf{x} + \mathbf{y}$ and $\mathbf{x} - \mathbf{y}$ are also orthogonal and their lengths are equal.

I.2 : Cross products

1. Compute the vector cross product $\mathbf{a} \times \mathbf{b}$, where $\mathbf{a} = (2, -3, 1)$ and $\mathbf{b} = (1, -2, 1)$.

2. Compute the vector cross product $\mathbf{a} \times \mathbf{b}$, where $\mathbf{a} = (12, -3, 0)$ and $\mathbf{b} = (-2, 5, 0)$.

3. Compute the vector cross product $\mathbf{a} \times \mathbf{b}$, where $\mathbf{a} = (1, 1, 1)$ and $\mathbf{b} = (2, 1, -1)$.

4. Compute the box product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$, where $\mathbf{a} = (2, 0, 1)$, $\mathbf{b} = (0, 3, 0)$ and $\mathbf{c} = (0, 0, 1)$.

5. Compute the triple cross products $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ for the vectors in the preceding exercise.

6. Compute the box product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$, where $\mathbf{a} = (1, 1, 0)$, $\mathbf{b} = (0, 1, 1)$ and $\mathbf{c} = (1, 0, 1)$.

7. Compute the triple cross products $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ for the vectors in the preceding exercise.

8. Compute the box product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$, where $\mathbf{a} = (1, 3, 1)$, $\mathbf{b} = (0, 5, 5)$ and $\mathbf{c} = (4, 0, 4)$.

9. Compute the triple cross products $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ for the vectors in the preceding exercise.

10. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are linearly independent vectors in $\mathbb{R}^3$, and that $\mathbf{c}$ is a nonzero vector which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Show that $\mathbf{c}$ is a scalar multiple of the cross product $\mathbf{a} \times \mathbf{b}$.

11. Suppose that $\mathbf{c}$ is a vector in $\mathbb{R}^3$, and define a mapping $\mathbf{D}$ from $\mathbb{R}^3$ to itself by the formula $\mathbf{Dv} = \mathbf{c} \times \mathbf{v}$. Verify that $\mathbf{D}$ is a linear transformation and that it satisfies the *Leibniz identity for products*: $\mathbf{D}(\mathbf{a} \times \mathbf{b}) = \mathbf{Da} \times \mathbf{b} + \mathbf{a} \times \mathbf{Db}$ **[*Hint:*** Use the Jacobi identity.**]**

I. 3 : Linear varieties

1. Let **L** and **M** be the lines in $\mathbb{R}^3$ consisting of all points expressible in the form **p** = **a** + t**b**, where

$$\mathbf{a} = (2, 3, 1) \text{ and } \mathbf{b} = (4, 0, -1) \text{ for the line } \mathbf{L}, \text{ and}$$

$$\mathbf{a} = (2, 3, 1) \text{ and } \mathbf{b} = (2, 2, 1) \text{ for the line } \mathbf{M}.$$

Determine whether **L** and **M** have a common point, and if so then find that point.

2. Let **L** and **M** be the lines in $\mathbb{R}^3$ consisting of all points expressible in the form **p** = **a** + t**b**, where

$$\mathbf{a} = (0, 2, -1) \text{ and } \mathbf{b} = (3, -1, 1) \text{ for the line } \mathbf{L}, \text{ and}$$

$$\mathbf{a} = (1, -2, -3) \text{ and } \mathbf{b} = (4, 1, -3) \text{ for the line } \mathbf{M}.$$

Determine whether **L** and **M** have a common point, and if so then find that point.

3. Let **L** and **M** be the lines in $\mathbb{R}^3$ consisting of all points expressible in the form **p** = **a** + t**b**, where

$$\mathbf{a} = (3, -2, 1) \text{ and } \mathbf{b} = (2, 5, -1) \text{ for the line } \mathbf{L}, \text{ and}$$

$$\mathbf{a} = (7, 8, -1) \text{ and } \mathbf{b} = (-2, 1, 2) \text{ for the line } \mathbf{M}.$$

Determine whether **L** and **M** have a common point, and if so then find that point.

4. Find the equation of the plane passing through the points $(0, 0, 0)$, $(1, 2, 3)$ and $(-2, 3, 3)$.

5. Find the equation of the plane passing through the points $(1, 2, 3)$, $(3, 2, 1)$ and $(-1, -2, 2)$.

6. Find the equation of the plane which passes through the point $(1, 2, 3)$ and is parallel to the $xy-$ plane.

7. Find the equation of the plane which contains the lines **L** and **M** given by all points expressible in the form

$$(1, 4, 0) + t(-2, 1, 1) \text{ for the line } \mathbf{L}, \text{ and}$$

$$(2, 1, 2) + t(-3, 4, -1) \text{ for the line } \mathbf{M}.$$

8. Find the line determined by the intersections of the two planes whose equations are $5x - 3y + z = 4$ and $x + 4y + 7z = 1$.

9. Let **L** and **M** be lines in $\mathbb{R}^2$ defined respectively by the linear equations $\mathbf{a}\cdot\mathbf{x} = b$ and $\mathbf{p}\cdot\mathbf{x} = q$. Show that if **L** and **M** are parallel (no points in common), then the two vectors **a** and **p** are linearly dependent.

10. Prove that the intersection of two linear varieties is a linear variety.

11. Let **H** and **K** be hyperplanes in $\mathbb{R}^n$, and assume that their intersection is nonempty. Prove that the intersection contains a line if n is at least 3. Furthermore, if n is at least 4 and **L** is a line in the intersection, prove that the latter also contains a point not on **L.** **[*Hint:*** The intersection is defined as the set of solutions of a system of two linear equations in n unknowns. Look at the set of solutions to the corresponding reduced system of equations.**]**

12. Let **S** and **T** be linear varieties in $\mathbb{R}^n$ which are defined by the systems of linear equations $\mathbf{a}_i \cdot \mathbf{x} = b_i$ and $\mathbf{c}_j \cdot \mathbf{x} = d_j$ respectively. Prove that their union **S** $\cup$ **T** is the set of all $\mathbf{x}$ such that $(\mathbf{a}_i \cdot \mathbf{x} - b_i)(\mathbf{c}_j \cdot \mathbf{x} - d_j) = 0$ for all i and j. **[*Hint:*** If $u \cdot v = 0$ in $\mathbb{R}$, then either $u = 0$ or else $v = 0$. As usual, there are two inclusions to verify.**]**

13. Let $\{\mathsf{P}_1, \mathsf{P}_2, \dots, \mathsf{P}_n\}$ be a finite set of points in $\mathbb{R}^3$, write each P_i in coordinate form as (a_i, b_i, c_i), and for each i let $\mathbf{q}_i = (a_i, b_i, c_i, 1)$. Prove that the points $\{\mathsf{P}_1, \mathsf{P}_2, \dots, \mathsf{P}_n\}$ are coplanar if and only if the vectors $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$ span a proper vector subspace of $\mathbb{R}^3$.

I.4 : Barycentric coordinates

1. Let **a**, **b**, and **c** be the noncollinear points in $\mathbb{R}^2$ whose coordinates are given by $(-1, 0)$, $(1, 0)$, and $(0, 1)$ respectively. Find the barycentric coordinates for each of the following points with respect to **a**, **b**, and **c.**

(*a*) $(0, 0)$ (*b*) $(1, 1)$

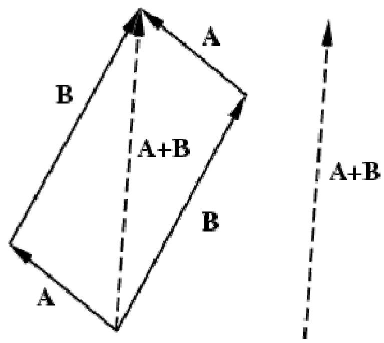
(*c*) $(\mathrm{sqrt}(2), \mathrm{sqrt}(2))$ (*d*) $(0, 5)$

(*e*) $(2, -1)$ (*f*) $(-\tfrac{1}{2}, -\tfrac{1}{3})$

2. Let $\mathbf{V}$ be a vector space over the real numbers. A subset $\{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n\}$ of $\mathbf{V}$ is said to be ***affinely independent*** if an arbitrary vector $\mathbf{w}$ in $\mathbf{V}$ has ***at most one*** expansion as a linear combination $\mathbf{w} = a_0\mathbf{v}_0 + a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n$ such that $a_0 + a_1 + \dots + a_n = 1$ (such expressions are often called ***affine combinations***). Prove that the set $\{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n\}$ is affinely independent if and only if the set $\{\mathbf{v}_1 - \mathbf{v}_0, \dots, \mathbf{v}_n - \mathbf{v}_0\}$ is linearly independent. Using the symmetry of the indices in the definition of affine independence, explain why the set $\{\mathbf{v}_1 - \mathbf{v}_0, \dots, \mathbf{v}_n - \mathbf{v}_0\}$ is linearly independent if and only if for each j the set of all nonzero vectors of the form $\mathbf{v}_i - \mathbf{v}_j$ (running over all i such that $i \neq j$) is linearly independent.

3. Suppose **a, b** and **d** are noncollinear points in $\mathbb{R}^2$. Prove that there is a unique point **c** distinct from **a, b** and **d** such that the lines **ab** and **cd** are parallel and the lines **ad** and **bc** are also parallel, and show that this unique point is given by $\mathbf{b} + \mathbf{d} - \mathbf{a}$. [***Hint:*** If **c** is given as above, note that $\mathbf{c} - \mathbf{d} = \mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{b} = \mathbf{d} - \mathbf{a}$, and let **V** and **W** be the $1-$ dimensional vector subspaces spanned by $\mathbf{b} - \mathbf{a}$ and $\mathbf{d} - \mathbf{a}$ respectively. Express all four lines in the form $\mathbf{x} + \mathbf{U}$ where **x** is one of the four points and **U** is one of **V** or **W.** What does the coset property imply if **ab** and **cd** have a point in common or if **ad** and **bc** have a point in common?]

Remark, The preceding exercise is closely related to the so – called "parallelogram law" for vector addition and reduces to the latter when $\mathbf{a} = \mathbf{0}$. In the figure below **A** and **B** correspond to $\mathbf{b} - \mathbf{a}$ and $\mathbf{d} - \mathbf{a}$, so that $\mathbf{A} + \mathbf{B}$ corresponds to $\mathbf{c} - \mathbf{a}$ and $\mathbf{d} = \mathbf{b} + \mathbf{c} - \mathbf{a}$.



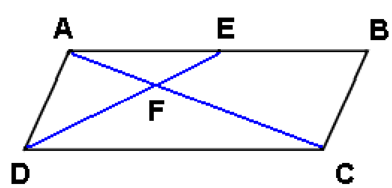
(***Source:*** http://mathworld.wolfram.com/ParallelogramLaw.html)

4. Suppose that the points **A, B, C, D** form the vertices of a parallelogram in $\mathbb{R}^2$, and let **E** be the midpoint of **A** and **B.** Prove that the lines **DE** and **AC** meet in a point **F** such that

(1) the distance from **A** to **F** is a third of the distance from **A** to **C,**

(2) the distance from **E** to **F** is a third of the distance from **E** to **D.**

Here is a picture that may be helpful in setting up a purely algebraic proof:



5. Suppose that we are given three noncollinear points **a, b, c** in $\mathbb{R}^2$, and suppose we are also given three arbitrary points in $\mathbb{R}^2$ with the following expansions in terms of barycentric coordinates:

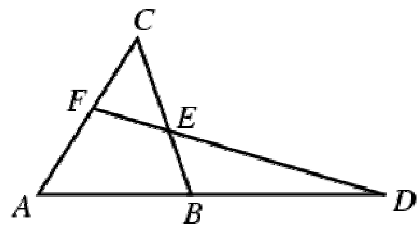
$$\begin{aligned} \mathbf{p}_1 &= t_1\mathbf{a} + u_1\mathbf{b} + v_1\mathbf{c} \\ \mathbf{p}_2 &= t_2\mathbf{a} + u_2\mathbf{b} + v_2\mathbf{c} \\ \mathbf{p}_3 &= t_3\mathbf{a} + u_3\mathbf{b} + v_3\mathbf{c} \end{aligned}$$

Show that the points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are collinear if and only if we have

$$\begin{vmatrix} t_1 & u_1 & v_1 \\ t_2 & u_2 & v_2 \\ t_3 & u_3 & v_3 \end{vmatrix} = \mathbf{0}.$$

6. Using the preceding exercise, prove the following result**,** which is essentially due to Menelaus of Alexandria ($\boldsymbol{c}$. 70 A. D. $-$ $\boldsymbol{c}$. 130 A. D.)**:**

Let **A, B, C** *be noncollinear points, and let* **D, E, F** *be points on the lines* **AB, BC** *and* **AC** *respectively. Express these points using barycentric coordinates as affine combinations* **D** $=$ t**A** $+$ $(1-t)$**B, E** $=$ u**B** $+$ $(1-u)$**C,** *and* **F** $=$ v**C** $+$ $(1-v)$**A.** *Then* **D, E** *and* **F** *are collinear if and only if* $tuv = -(1-t)(1-u)(1-v)$**.**



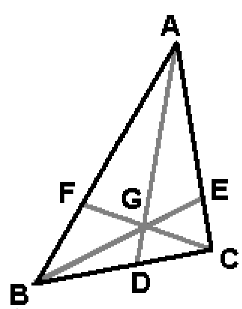
(***Source:*** **http://mathworld.wolfram.com/MenelausTheorem.html**)

7. In the preceding exercise**,** suppose that **D, E** and **F** are collinear such that **B** is halfway between **A** and **D,** while **E** is halfway between **B** and **C.** Express the vector **F** as a linear combination of **A** and **C.**

8. Using Exercise **5,** prove the following result due to G. Ceva (1647 – 1734)**:**

Let **A, B, C** *be noncollinear points, let* **D, E, F** *be points on the lines* **BC, AC** *and* **AB** *respectively such that* {**D, E, F**} *and* {**A, B, C**} *are disjoint, and suppose that the lines* **BE** *and* **CF** *intersect at some point* **G** *which is not equal to* **B** *or* **C.** *Express the points* **D, E, F** *in terms of barycentric coordinates as* **D** $=$ t**B** $+$ $(1-t)$**C,** **E** $=$ u**C** $+$ $(1-u)$**A,** *and* **F** $=$ v**A** $+$ $(1-v)$**B.** *Then the lines* **AD, BE** *and* **CF** *are concurrent (in other words, the three lines have a point in common) if and only if we have*

$$tuv = (1-t)(1-u)(1-v).$$



(***Source:*** **http://mathworld.wolfram.com/CevasTheorem.html**)

[*Hint:* The lines **AD, BE** and **CF** are concurrent if and only if the points **A, D** and **G** are collinear.**]**

9. In the setting of the preceding exercise, suppose that the three lines **AD, BE** and **CF** are concurrent with $t = \frac{1}{2}$ and $v = 1 - u$. Express the common point **G** of these lines as a linear combination of **A** and **D** with the coefficients expressed in terms of u.

10. Let $\mathbf{V}$ be a vector space over the real numbers, let $\mathbf{S} = \{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a subset of $\mathbf{V}$, and let $\mathbf{T} = \{\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_m\}$ be a set of vectors in $\mathbf{V}$ which are affine combinations of the vectors in $\mathbf{S}$. Suppose that $\mathbf{y}$ is a vector in $\mathbf{V}$ which is an affine combination of the vectors in $\mathbf{T}$. Prove that $\mathbf{y}$ is also an affine combination of the vectors in $\mathbf{S}$.

11. Find the barycentric coordinates of the point $(2, 0)$ with respect to the noncollinear points $(1, 0)$, $(3, 1)$, and $(3, -1)$.