Math 133
Winter 2009

MORE EXERCISES FOR SECTIONS III.1 AND III.2

Definition. If V is a vector subspace of $\mathbb{R}^n$ then its *orthogonal complement* $V^\perp$ is the set of all $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x} \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in V$.

D1. Suppose that V is an r-dimensional vector subspace of $\mathbb{R}^n$. Prove that the orthogonal complement $V^\perp$ is an $(n-r)-$dimensional vector subspace of $\mathbb{R}^n$ and that $\left(V^\perp\right)^\perp = V$. [*Hint:* Take an orthonormal basis B for V, extend it to a basis for $\mathbb{R}^n$ by adding a suitable set of vectors A, and use the Gram-Schmidt process to find an orthonormal basis C of $\mathbb{R}^n$ containing B. Show that the set D of all vectors in C but not B must be an orthonormal basis for $V^\perp$. For the final assertion show that V is a vector subspace of $\left(V^\perp\right)^\perp = V$ and the dimensions of these two subspaces are equal.]

D2. Suppose that V and W are respectively $1-$ and $2-$dimensional vector subspaces of $\mathbb{R}^3$ such that $V \cap W = \{\mathbf{0}\}$ but $V \neq W^\perp$. Prove that $V^\perp \cap W$ is a $1-$dimensional vector subspace of $\mathbb{R}^3$.

D3. Let L and P be a line and plane in $\mathbb{R}^3$ which meet at a point $\mathbf{x}$, and assume that L is not perpendicular to P. Prove that there is a unique line M such that $\mathbf{x} \in M \subset P$ and $L \perp M$.

D4. Suppose that we are given positive numbers a and x. Prove that there is an isosceles triangle ΔABC with $d(B,C) = a$ and $d(A,B) = d(B,C) = x$ if and only if $2x > a$. [*Hint:* For one direction use the Triangle Inequality, and for the other direction show that if $2x > a$ then there is right triangle whose hypotenuse has length x and one of whose other sides has length $a/2$. How can we use this to construct an isosceles triangle with the desired measurements?]