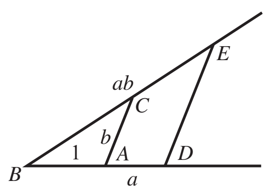
the corresponding vertices all pass through the same point. As the principal creator of projective geometry as a separate mathematical discipline, Poncelet can rightly be called the "father of modern geometry."

## 8.2 Problems

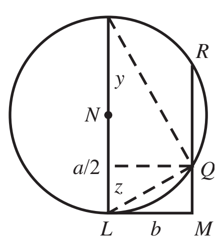
**1.** To multiply two numbers geometrically, Descartes said:

> Let $AB$ be taken as unity, and let it be required to multiply $BD$ by $BC$. I have only to join the points $A$ and $C$, and draw $DE$ parallel to $CA$; then $BE$ is the product of $BD$ and $BC$.

Show that the length of $BE$ is the product of the lengths of $BD$ and $BC$.

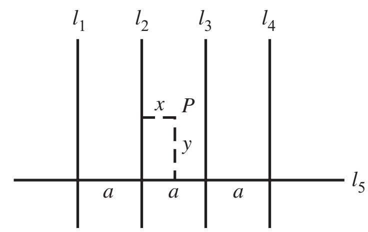



**2.** In *La Géométrie*, Descartes constructed the positive solutions to the quadratic equation $x^2 = ax - b^2$, where $b < a/2$. Given a circle of radius $NL = a/2$, draw a tangent to $L$ and lay off from the point of contact a length $LM = b$. Then, through $M$, draw a line parallel to $NL$,




cutting the circle in the points $Q$ and $R$. Prove that the lengths $MQ$ and $MR$ represent the two positive solutions to $x^2 = ax - b^2$. [*Hint:* If the parallel to $LM$ through $Q$ cuts the diameter in segments of length $y$ and $z$, then $y + z = a$ and $yz = b^2$.]

**3.** Assume that in the five-line Pappus problem, four of the lines $l_1$, $l_2$, $l_3$, and $l_4$ are parallel and an equal distance apart, and that the fifth line $l_5$ is perpendicular to the others. Prove that if $l_5$ and $l_2$ are taken as the $x$-axis and $y$-axis,




respectively, and if $p_k$ denotes the distance of a point $P = (x, y)$ from the line $l_k$, then the locus of all points satisfying $p_1 p_3 p_4 = a p_2 p_5$ is given by

$$(a + x)(a - x)(2a - x) = axy.$$

This locus, which occurs in *La Géométrie*, was later called the Cartesian parabola, or trident, by Newton.

**4.** Show that the equation $x^3 - x^2 + 2x + 1 = 0$ has no positive roots. [*Hint:* Multiply by $x + 1$, which does not change the number of positive roots.]

**5.** Find the number of positive roots of the equation $x^5 + 2x^3 - x^2 + x - 1 = 0$.

**6.** From Descartes's rule of signs, conclude that the equation $x^{2n} - 1 = 0$ has $2n - 2$ imaginary roots.

**7.** Without actually obtaining these roots, show that

(a) $x^3 + 3x + 7 = 0$ and
(b) $x^6 - 5x^5 - 7x^2 + 8x + 20 = 0$

both possess imaginary roots.

**8.** Verify the following assertions.

(a) If all the coef cients of an equation are positive and the equation involves no odd powers of $x$, then all its roots are imaginary.
(b) If all the coef cients of an equation are positive and all terms involve odd powers of $x$, then zero is the only real root of the equation.
(c) An equation with only positive coef cients cannot have a positive root.

**9.** Prove that

(a) The equation $x^3 + a^2 x + b^2 = 0$ has one negative and two imaginary roots if $b \neq 0$.