~

Mathematics 133, Winter 2009, Examination 3

Answer Key

1. [*20 points*] Assume we are working inside the **Euclidean coordinate plane.**

Let $h > 0$, and consider the isosceles triangle whose vertices are $A = (-1, 0)$, $B = (1, 0)$ and $C = (0, h)$. Find the (coordinates of) the circumcenter for ΔABC. [*Hint:* First explain why the circumcenter lies on the y-axis, which is the perpendicular bisector of $[AB]$, and use this to simplify the computations.]

SOLUTION.

The circumcenter lies on the perpendicular bisectors of the three sides of ΔABC, and the perpendicular bisector of $[AB]$ is the y-axis, so the circumcenter must lie on the y-axis and its x-coordinate is zero.

One way to find the perpendicular bisector of $[BC]$ is to think of it as the set of all points equidistant from B and C, so that its equation is

$$(x-1)^2 \;+\; y^2 \quad = \quad x^2 \;+\; (y-h)^2$$

which simplifies to

$$1 \;-\; 2x \quad = \quad h^2 \;-\; 2yh \;.$$

The circumcenter's coordinates satisfy this equation and $x = 0$. If we solve these two equations for x and y, we see that

$$y \quad = \quad \frac{h^2 - 1}{2h} \;.$$

2. [*20 points*] Assume we are working inside some **Euclidean** plane.

Suppose that we are given ΔABC and ΔDEF such that $\Delta ABC \sim \Delta DEF$, and let $G \in (BC)$ and $H \in (EF)$ be the feet of perpendiculars from A to BC and D to EF respectively. Using the basic similarity theorems for triangles, prove that

$$\frac{d(D,H)}{d(A,G)} = \frac{d(E,F)}{d(B,C)}$$

(in other words, the lengths of the altitudes of the two triangles are proportional to the lengths of the sides).

SOLUTION.

We know that $\angle ABG = \angle ABC$ and $\angle DEF = \angle DEH$, and therefore the assumption that $\Delta ABC \sim \Delta DEF$ implies that $|\angle ABG| = |\angle ABC| = |\angle DEF| = |\angle DEH|$. Since AG and DH are altitudes, we also know that $|\angle AGB| = 90^\circ = |\angle DHE|$. Therefore $\Delta AGB \sim \Delta DHE$ by the **AA** Similarity Theorem. By the definition of similar triangles this yields the proportionality equation

$$\frac{d(D,E)}{d(A,B)} = \frac{d(D,H)}{d(A,G)} .$$

But we also have $\Delta ABC \sim \Delta DEF$, which yields the proportionality equation

$$\frac{d(D,E)}{d(A,B)} = \frac{d(E,F)}{d(B,C)} .$$

Combining these, we obtain the ratio equation stated in the exercise.

NOTE. Drawings for this problem and further discussion appear in the online file `exam3w09comments.pdf`.

3. [*15 points*] Assume we are working inside some **Euclidean** plane.

Suppose that in ΔABC we have $d(A,B) = 2$, $d(B,C) = 3$ and $d(A,C) = 4$. Let D be the point on (BC) such that $[AD$ bisects $\angle BAC$. Find $d(B,D)$ and $d(C,D)$.

SOLUTION.

Let $x = d(B,D)$, so that $d(C,D) = 3 - x$ (because we have $B * D * C$).

By the Angle Bisector Theorem (Theorem III.5.13 in the notes) we have

$$\frac{2}{4} \quad = \quad \frac{d(A,B)}{d(A,C)} \quad = \quad \frac{d(B,D)}{d(C,D)} \quad = \quad \frac{x}{3-x}$$

so that $2(3-x) = 4x$. This simplifies to $6 - 2x = 4x$, which means that $x = 1$. Therefore we have $d(B,D) = x - 1$ and $d(C,D) = 3 - x = 2$.

4. [*20 points*] Assume we are working inside some **hyperbolic** plane.

Suppose that in ΔABC is an isosceles triangle with $d(A,B) = d(A,C)$, and let D and E be points satisfying the conditions $A * B * D$, $A * C * E$, and $d(B,D) = d(C,E)$. Prove that $|\angle ADE| \neq |\angle ABC|$ and determine which of these is smaller.

SOLUTION.

This is essentially Exercise V.4.5, with one small twist.

Since we have $A * B * D$ and $A * C * E$ hold, we have

$$d(A,D) \;=\; d(A,B) + d(B,D) \;=\; d(A,C) + d(C,E) \;=\; d(A,E)$$

so that ΔADE is isosceles. Therefore $|\angle ADE| = |\angle AED|$ by the Isosceles Triangle Theorem; note that the assumption in the exercise and the same theorem also imply that $|\angle ABC| = |\angle ACB|$. Using these and the fact that $\angle DAE = \angle BAC$, we may write the angle defects of ΔABC and ΔADE as follows:

$$\delta(\Delta ABC) \;=\; 180^\circ - |\angle BAC| - 2 \cdot |\angle ABC|$$

$$\delta(\Delta ADE) \;=\; 180^\circ - |\angle DAE| - 2 \cdot |\angle ADE|$$

Since $B \in (AD)$ and $C \in (AE)$, two applications of Theorem V.4.4 imply that $\delta(\Delta ADE) > \delta(\Delta ABE) > \delta(\Delta ABC)$, so that

$$180^\circ - |\angle DAE| - 2 \cdot |\angle ADE| \;>\; 180^\circ - |\angle BAC| - 2 \cdot |\angle ABC| \;.$$

Since $\angle ADE = \angle ABC$, this inequality implies that $|\angle ADE| < |\angle ABC|$.

NOTES. In Euclidean geometry one has a different conclusion; namely, $|\angle ADE| = |\angle ABC|$. A drawing for this problem and further discussion appear in the online file `exam3w09comments.pdf`.

5. [*10 points*] Assume we are working inside some **Euclidean** plane.

(a) Given ΔABC, define its centroid, incenter and orthocenter.

(b) Which (if any) of the points in (a) are in the interior of ΔABC for all triangles ΔABC, and which (if any) do not necessarily lie in the interior of ΔABC?

SOLUTION.

(a) The **centroid** is the point where the three medians of the triangle (the lines joining the vertices to the midpoints of the opposite side) meet. The **incenter** is the point where the three angle bisectors of the vertex angles meet. The **orthocenter** is the point where the three altitudes (perpendiculars from the vertices to the opposite sides) meet.

(b) The centroid and incenter always lie in the interior of the triangle but the orthocenter does not. (If all three vertex angles are acute, then the orthocenter lies in the interior, if one vertex angle is a right angle then the orthocenter is the corresponding vertex, and if one vertex angle is obtuse, then the orthocenter lies in the exterior; as suggested by the drawings for Problem 2 in `exam3w09comments.pdf`, in the third case an altitude to either edge of the obtuse angle contains no points in the interior of the triangle and meets the triangle in only one point — the vertex through which it passes.)

6. [*15 points*] Assume we are working inside some **neutral** plane.

(a) State the Saccheri-Legendre Theorem.

(b) Suppose that A, B, C, D are four points such that no three are collinear. State the defining conditions for A, B, C, D (in that order) to be the vertices of a Saccheri quadrilateral with base $[AB]$.

(c) Suppose that A, B, C, D are four points such that no three are collinear. State the defining conditions for A, B, C, D (in that order) to be the vertices of a Lambert quadrilateral.

SOLUTION.

(a) Given ΔABC, we have $|\angle ABC| + |\angle BCA| + |\angle CAB| \leq 180°$.

(b) The points C and D must lie on the same side of AB, with AB perpendicular to BC and AD and $d(A, D) = d(B, C)$ (equivalently, the vertices form the vertices of a convex quadrilateral plus the last two conditions).

(c) Three (or more) of the angles $\angle ABC$, $\angle BCD$, $\angle CDA$, $\angle DAB$ must be right angles.