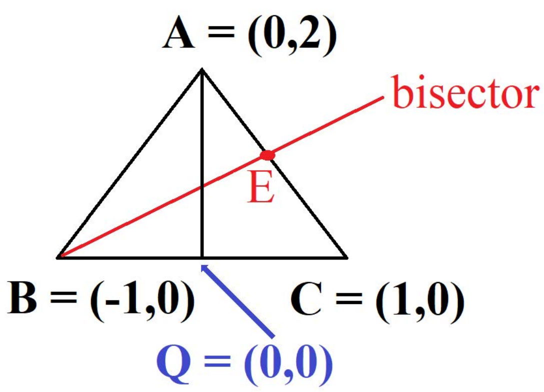
Math 133, Fall 2018, Quiz 3

Suppose we are given isosceles triangle ABC in the coordinate plane, where $A = (0,2)$, $B = (-1,0)$ and $C = (1,0)$; denote $(0,0)$ by Q. Let $E \in (AC)$ be chosen such that [BE bisects $\angle ABC$. Find the length $|AE|$.



Solution. By Theorem III.5.13 in the course notes, we know that

$$\frac{|BA|}{|BC|} = \frac{|AE|}{|EC|} .$$

Furthermore, since $AQ \perp BC$ by construction, by the Pythagorean Theorem we have

$$|AB|^2 = |AQ|^2 + |BQ|^2 = 1 + 4 = 5$$

and by construction we clearly also have $|BC| = 2$. This yields the following equation:

$$\frac{|AE|}{|EC|} = \frac{\sqrt{5}}{2}$$

Now we also have $|BC| = \sqrt{5}$ because ΔABC is isosceles, and furthermore $A * E * C$ implies that $|AE| + |EC| = \sqrt{5}$. If we set $x = |AE|$, we then have the following equation for x:

$$\frac{x}{\sqrt{5} - x} = \frac{\sqrt{5}}{2}$$

If we multiply both sides by the product of the denominators, we obtain a linear equation in x:

$$2x = \sqrt{5}\left(\sqrt{5} - x\right)$$

Since $(\sqrt{5}+2)(\sqrt{5}-2) = 1$, the solution to this equation is $x = 5(\sqrt{5}-2)$, and by construction the right hand side is also equal to $|AE|$. Since $\sqrt{5}-2$ is well within .01 per cent of the approximation 0.2360679774997896964, it follows that $|AE|$ is very close to 1.180339887498948432.∎